

Dynamic Pricing in High-dimensions

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We study the pricing problem faced by a firm that sells a large number of products, described via a wide range of features, to customers that arrive over time. This is motivated in part by the prevalence of online marketplaces that allow for real-time pricing. We propose a dynamic policy, called Regularized Maximum Likelihood Pricing (RMLP), that obtains asymptotically optimal revenue. Our policy leverages the structure (sparsity) of a high-dimensional demand space in order to obtain a logarithmic regret compared to the clairvoyant policy that knows the parameters of the demand in advance. More specifically, the regret of our algorithm is of $O(s_0 \log T(\log d + \log T))$, where d and s_0 correspond to the dimension of the demand space and its sparsity. Furthermore, we show that no policy can obtain regret better than $O(s_0(\log d + \log T))$.

Key words: Revenue Management, Dynamic Pricing, High-dimensional Regression, Maximum Likelihood, Sparsity, Hypothesis Testing

1. Introduction

A central challenge in revenue management is determining the optimal pricing policy when there is uncertainty about customers' willingness to pay. Due to its importance, this problem has been studied extensively (Kleinberg and Leighton 2003, Besbes and Zeevi 2009, Badanidiyuru et al. 2013, Wang et al. 2014, Broder and Rusmevichientong 2012, Keskin and Zeevi 2014, den Boer and Zwart 2014, Cohen et al. 2016). Most of these models are built around the following classic setting: customers arrive over time; the seller posts a price for each customer; if the customer's valuation is above the posted price, he will purchase the product, otherwise, he will leave; based on this and the previous feedback, the seller updates the posted price. Hence, the seller is involved in the realm of exploration-exploitation as he needs to choose between learning about the valuations (demand curve) and exploiting what has been learned so far to collect revenue.

In this work, we consider a setting with a large number of products which are defined via a wide range of features. We focus on a single customer and study a model where valuations of the customer is given by $v(\theta, x)$ with x being the (observable) feature vectors of products and θ representing the true parameters of the "demand curve", which is a-priori unknown

to the seller; see Cohen et al. (2016). An important special case of this setting is the linear model in which $v(\theta, x) = \theta \cdot x + \epsilon$ where ϵ captures the idiosyncratic noise in valuations.

Our setting is motivated in part by applications in online marketplaces. For instance, a company such as Airbnb recommends prices to hosts based on many features including the space (number of rooms, beds, bathrooms, etc.), amenities (AC, WiFi, washer, parking, etc.), the location (accessibility to public transportation, walk score of the neighborhood, etc.), house rules (pet-friendly, non-smoking, etc.), as well as the prediction of the demand which itself depends on many factors including the date, events in the area, availability and prices of near-by hotels, etc (Airbnb Documentation 2015). Therefore, the feature vector describing each property can have hundreds of features. Another important application comes from online advertising. Online publishers set the (reserve) price of ads based on many features including user’s demographic, browsing history, the context of the webpage, the size and location of the ad on the page, etc.

We propose a dynamic policy called *Regularized Maximum Likelihood Pricing* (RMLP). As suggested by its name, the policy uses maximum likelihood method to estimate the true parameters of the demand curve. In addition, using an (ℓ_1 -norm) regularizer, our policy exploits the structure of the optimal solution, namely, its performance significantly improves if the valuations are essentially determined by a small subset of features. More formally, the difference between the revenue obtained by our policy and the benchmark policy that knows the true parameters of the demand curve θ , in advance, is bounded by $O(s_0 \log T (\log d + \log T))$, where T , d , and s_0 respectively denote the length of the horizon, number of the features, and sparsity (i.e, number of non-zero elements of the true demand curve). We show that our results are tight up to a logarithmic factor. Namely, no policy can obtain regret better than $O(s_0 (\log d + \log T))$.

We point out that our results can be applied to (many) applications where the features dimensions are larger than the time horizon of interest. A powerful pricing policy for these applications should obtain regret that scales gracefully with the dimension. Note that in general, little can be learned about θ if $T < d$, because the number of degrees of freedom d exceeds the number of observations T , and therefore, any estimator can be arbitrary erroneous. However, when there is prior knowledge about the unknown parameter θ , e.g. that has *sparsity* structure, then accurate estimations are attainable even when $T < d$.

Organization. The rest of the paper is organized as follows: First, we discuss how our work is positioned with respect to the literature. In Section 2, we formally present our model and discuss the technical assumptions and the benchmark policy. The RMLP policy is presented in Section 3, followed by its analysis in Section 4. We provide in Section 5, a lower-bound on the performance of any dynamic pricing policy that does not know the demand curve in advance. In Section 6, we generalize the RMLP policy to non-linear valuations functions. The proofs are relegated to the appendix.

Related Work

We list below a few lines of research on dynamic pricing and statistical learning that are related to our work.

Dynamic Pricing and Learning. The literature on dynamic pricing and learning has been growing over the past few years, motivated in part by the advances in big data technology that allow firms to easily collect and use data. We briefly discuss some of the recent lines of research in this literature. We refer to den Boer (2015) for an excellent survey on this topic.

Parametric Approach. A natural approach to capture uncertainty about the customers' valuations is to model the uncertainty using a small number of parameters, and then estimate those parameters using classical statistical methods such as maximum likelihood (Broder and Rusmevichientong 2012, den Boer and Zwart 2013, 2014) or least square estimation (Goldenshluger and Zeevi 2013, Keskin 2014, Bastani and Bayati 2016). Our work is similar to this line of work, in that we assume a parametric model for customer's valuations and apply the maximum likelihood method using the randomness of the idiosyncratic noise in valuations. However, the parameter vector θ is high-dimensional, whose dimension d can exceed the time horizon of interest T . We use *regularized* maximum-likelihood in order to promote sparsity structure in the estimated parameter. Further, our pricing policy has an episodic theme which makes the posted prices p_t in each episode independent of the idiosyncratic noise in valuations, z_t , in that episode. This is in contrast to other policies based on maximum-likelihood, such as MLE-GREEDY (Broder and Rusmevichientong 2012), or greedy iterative least square (GILS) (Keskin 2014, den Boer and Zwart 2014, Bastani and Bayati 2016) that use the entire history of observations to update the estimate for the model parameters at each step.

Bayesian Approach. One of the earliest work on Bayesian parametric approach in this context is by Rothschild (1974) who consider a Bayesian framework where the firm can choose from two prices with unknown demand and show that (myopic) Bayesian policies may lead to “incomplete learning.” However, carefully designed variations of the myopic policies can (optimally) learn the optimal price (Harrison et al. 2012); see also Keller and Rady (1999), Araman and Caldentey (2009), Farias and Van Roy (2010), Keskin and Zeevi (2014).

Non-Parametric models. An early work in non-parametric setting is by Kleinberg and Leighton (2003). They model the dynamic pricing problem as a multi-armed bandit (MAB) where each arm corresponds to a (discretized) posted price. They propose an $O(\sqrt{T})$ -algorithm where T is the length of the horizon. Similar results have been obtained in more general settings (Badanidiyuru et al. 2013, Agrawal and Devanur 2014) including setting with inventory constraints (Besbes and Zeevi 2009, Babaioff et al. 2012, Wang et al. 2014).

Feature-based Models. Recent papers on dynamic pricing consider models with features/covariates. Amin et al. (2014), in a model similar to ours, present an algorithm that obtains regret $O(T^{2/3})$; they also study dynamic incentive compatibility in repeated auctions. Another closely related work to ours is by Cohen et al. (2016). Their model differs from ours in two main aspects: *i*) their model is deterministic (no idiosyncratic noise) *ii*) the arrivals (of features vectors) is modeled as adversarial. They propose a clever binary-search approach using the Ellipsoid method which obtains regret of $O(d^2 \log(T/d))$. Qiang and Bayati (2016) study a model where the seller can observe the demand itself, not a binary signal as in our setting. They show that a myopic policy based on least-square estimations can obtain a logarithmic regret. To the extent of our knowledge, ours is the first work that highlights the role of structure/sparsity in dynamic pricing.

Bastani and Bayati (2016) study a multi-armed bandit setting, with discrete arms, and high-dimensional covariates, generalizing results of Goldenshluger and Zeevi (2013). Bastani and Bayati (2016) present an algorithm, using a LASSO estimator, that obtains regret $O(K(\log T + \log d)^2)$ where K denotes the number of arms. In contrast, our setting can be interpreted as a multi-armed bandit with continuous arms in a high dimensional space.

High Dimensional Statistics. There has been a great deal of work on regularized estimator under the high-dimensional scaling; see Van de Geer (2008), Ravikumar et al. (2010), Bunea et al. (2008), Kakade et al. (2010), Negahban et al. (2012) for a non-exhaustive list. The most active area in this context is perhaps sparse linear regression, where many of the impactful ideas in high dimensional inference have been developed (Candes and Tao 2005, Donoho 2006, Van De Geer et al. 2009, Bickel et al. 2009, Candes and Tao 2007, Meinshausen and Yu 2009, Wainwright 2009). Focus of theoretical results has been on establishing order optimal guarantees on prediction error, estimation error of the parameters, as well as variable selection. Some of these results have also been extended to the setting of generalized linear model (GLM). For instance, Negahban et al. (2012) considers GLMs where conditional on a feature vector x , response y has distribution of form $\mathbb{P}_\theta(y|x) \propto \exp \left\{ \frac{y(x \cdot \theta) - \Phi(x \cdot \theta)}{c(\sigma)} \right\}$, where $c(\sigma)$ is a fixed known normalization parameter and function Φ is the link function. It is shown that these GLM-based models satisfies a form of restricted strong convexity which makes them amenable to analysis, and the estimation ℓ_2 bounds are derived for such model.

Closer to the spirit of our work is the problem of 1-bit compressed sensing (Plan and Vershynin 2013a, Bhaskar and Javanmard 2015, Plan and Vershynin 2013b, Jacques et al. 2013, Ai et al. 2014). In this problem, linear measurements are observed for an unknown parameter of interest but only the sign of these measurements are observed. Note that in our problem, seller is involved in both the learning task and also the policy design. Specifically, he should decide on the prices, which directly affect collected revenue and also indirectly influence the difficulty of the learning task. The market values are then compared with the posted prices, in contrast to 1-bit compressed sensing where the measurements are compared with zero (sign information). In addition, the pricing problem has an online nature while the 1-bit compressed sensing is mostly studied for offline setting. Finally, note that since the posted prices depend on previous observations, they bring in dependency between samples which is undesired for learning task.

2. Model

We consider a seller, who has a product for sale in each period $t = 1, 2, \dots, T$, where T denotes the length of the horizon and may be unknown to the seller. Each product is represented by an *observable* vector of features (covariates) $x_t \in \mathcal{X} \subseteq \mathbb{R}^d$. Products may

vary across periods and we assume that feature vectors x_t are sampled independently from a fixed, but a-priori *unknown*, distribution \mathbb{P}_X , supported on a bounded set \mathcal{X} .

The product at time t has a market value $v_t = v(x_t)$, which is *not observed* by the seller and function v is (a-priori) unknown. At each period t , the seller posts a price p_t . If $p_t \leq v_t$, a sale occurs, and the seller collects revenue p_t . If the price is set higher than the market value, $p_t > v_t$, no sale occurs and no revenue is obtained. The goal of the seller is to design a pricing policy that maximizes the collected revenue.

We assume that the market value of a product is a linear function of its covariates, namely

$$v(x_t) = \theta^* \cdot x_t + z_t, \quad (1)$$

where $a \cdot b$ denotes the inner product of vectors a and b . Here, $\{z_t\}_{t \geq 1}$ are idiosyncratic shocks, referred to as noise, which are drawn independently and identically from a distribution with mean zero and cumulative function F , with density $f(x) = F'(x)$. The noise can account for the features that are not measured. We partly generalize our model to non-linear valuation functions in Section 6.

Parameter θ^* is a-prior unknown to seller. Therefore, the seller is involved in the realm of exploration-exploitation as he needs to choose between learning θ^* and exploiting what has been learned so far to collect revenue.

Let y_t be the response variable that indicates whether a sale has occurred at period t :

$$y_t = \begin{cases} +1 & \text{if } v_t \geq p_t, \\ -1 & \text{if } v_t < p_t. \end{cases} \quad (2)$$

Note that the above model can be represented as the following probabilistic model:

$$y_t = \begin{cases} +1 & \text{with probability } 1 - F(p_t - \theta^* \cdot x_t), \\ -1 & \text{with probability } F(p_t - \theta^* \cdot x_t) \end{cases} \quad (3)$$

Our proposed algorithm exploits the structure (sparsity) of the feature space to improve its performance, namely, if only a few of the covariates are predictor of the market value. Let s_0 be the number of nonzero coordinates of θ^* , $s_0 = \|\theta^*\|_0 = \sum_{j=1}^d \mathbb{I}(\theta_j^* \neq 0)$, which is of course a-priori unknown to the seller.

2.1. Technical Assumptions

To simplify the presentation, we assume that $\|x_t\|_\infty \leq 1$, for all $x_t \in \mathcal{X}$, and $\|\theta^*\|_1 \leq W$ for a known constant W , where for a vector $u = (u_1, \dots, u_d)$, $\|u\|_\infty = \max_{i \in [d]} |u_i|$ denotes the maximum absolute value of its entries and $\|u\|_1 = \sum_{i=1}^d |u_i|$. We denote by Ω the set of feasible θ^* , i.e., $\Omega = \{\theta \in \mathbb{R}^d : \|\theta\|_0 \leq s_0\} \cap \{\theta \in \mathbb{R}^d : \|\theta\|_1 \leq W\}$.

We also make the following assumption on the distribution of noise F .

ASSUMPTION 1. *The function $F(v)$ is strictly increasing. Further, $F(v)$ and $1 - F(v)$ are log-concave in v .*

Log-concavity is a widely-used assumption in the economics literature (Bagnoli and Bergstrom 2005). Note that if the density f is symmetric and the distribution F is log-concave, then $1 - F$ is also log-concave. Assumption 1 is satisfied by several common probability distributions including normal, uniform, Laplace, exponential, and logistic. Note that the cumulative distribution function of all log-concave densities is also log-concave (Boyd and Vandenberghe 2004).

We also need an assumption on the distribution of the feature vectors.

ASSUMPTION 2. *Assume that the distribution of covariates, \mathbb{P}_X has a bounded support \mathcal{X} . In addition, there exist constants C_{\min} and C_{\max} such that for every eigenvalue σ of its covariance matrix Σ , we have $0 < C_{\min} \leq \sigma < C_{\max} < \infty$.*

A generic example of a probability distribution with positive definite covariance is the case that \mathbb{P}_X is bounded below from zero on an open set around the origin. In particular, $\mathbb{P}_X \geq a$ for all $\|x\|_\infty \leq c$ for some constants $a, c > 0$. This condition holds for many common probability distributions, such as uniform, truncated normal, and in general truncated version of many more distributions.

2.2. Benchmark Policy and Regret Minimization

We evaluate the performance of our algorithm using the common notion of regret: the expected revenue loss compared with the optimal pricing policy that knows θ^* in advance (but not the realizations of $\{z_t\}_{t \geq 1}$). Let us first characterize this benchmark policy.

Using Eq. (1), the expected revenue from a posted price p is equal to $p \times \mathbb{P}(v_t \geq p) = p(1 - F(p - \theta^* \cdot x_t))$. Therefore, using first order conditions, for the optimal posted price, denoted by p^* , we have

$$p^*(x_t) = \frac{1 - F(p^* - \theta^* \cdot x_t)}{f(p^* - \theta^* \cdot x_t)}. \quad (4)$$

To simplify the presentation, let $p_t^* = p^*(x_t)$ denote the optimal price at time t .

We now define $\varphi(v) \equiv v - \frac{1-F(v)}{f(v)}$ corresponding to the *virtual valuation* function commonly used in mechanism design (Myerson 1981). By Assumption 1, φ is injective and hence we can define function g as follows

$$g(v) \equiv v + \varphi^{-1}(-v). \quad (5)$$

It is easy to verify that g is non-negative. Note that by Eq. (4), for the optimal price we have

$$\theta^* \cdot x_t + \varphi(p^* - \theta^* \cdot x_t) = 0.$$

Therefore, by rearranging the terms for the optimal price at time t we have

$$p_t^* = g(\theta^* \cdot x_t). \quad (6)$$

We can now formally define the regret of a policy. Let π be the seller's policy that sets price p_t at period t , and p_t can depend on the history of events up to time t . The worst-case regret is defined as:

$$\text{Regret}_\pi(T) \equiv \max_{\substack{\theta^* \in \Omega \\ \mathbb{P}_X \in Q(\mathcal{X})}} \mathbb{E} \left[\sum_{t=1}^T \left(p_t^* \mathbb{I}(v_t \geq p_t^*) - p_t \mathbb{I}(v_t \geq p_t) \right) \right], \quad (7)$$

where the expectation is with respect to the distributions of idiosyncratic noise, z_t , and \mathbb{P}_X , the distribution of feature vectors. Moreover, $Q(\mathcal{X})$ represents the set of probability distributions supported on a bounded set \mathcal{X} .

Our algorithm uses the sparsity structure of θ^* and learns the model with order of magnitude less data compared to a structure-ignorant algorithm. In Section 4, we show that our pricing scheme achieves a regret bound of $O(s_0 \log T (\log d + \log T))$.

3. A Regularized Maximum Likelihood Pricing (RMLP) Policy

In this section, we present our dynamic pricing policy. Our policy runs in an episodic fashion. Episodes are indexed by k and time periods are indexed by t . The length of episode k is denoted by τ_k . Throughout episode k , we set the prices equal to $p_t = g(\langle x_t, \hat{\theta}^k \rangle)$ where $\hat{\theta}^k$ denotes the estimate of θ^* which is obtained from the observations $\{(x_t, y_t, p_t)\}$ in the *previous* episode. Note that by Eq. (5), p_t is the optimal posted price if $\hat{\theta}^k$ was the true underlying parameter of the model.

Input: (at time 0) function g , regularizations λ_k , W (bound on $\|\theta^*\|_1$),

Input: (arrives over time) covariate vectors $\{x_t\}_{t \in \mathbb{N}}$

Output: prices $\{p_t\}_{t \in \mathbb{N}}$

1: $\tau_1 \leftarrow 1$, $p_1 \leftarrow 0$, $\hat{\theta}^1 \leftarrow 0$

2: **for** each episode $k = 2, 3, \dots$ **do**

3: Set the length of k -th episode: $\tau_k \leftarrow 2^{k-1}$.

4: Update the model parameter estimate $\hat{\theta}^k$ using the regularized ML estimator obtained from observations in the previous episode:

$$\hat{\theta}^k = \arg \min_{\|\theta\|_1 \leq W} \{\mathcal{L}(\theta) + \lambda_k \|\theta\|_1\} \quad (8)$$

with

$$\mathcal{L}(\theta) = -\frac{1}{\tau_{k-1}} \sum_{t=\tau_{k-1}}^{\tau_k-1} \left\{ \mathbb{I}(y_t = 1) \log(1 - F(p_t - \theta \cdot x_t)) + \mathbb{I}(y_t = -1) \log(F(p_t - \theta \cdot x_t)) \right\} \quad (9)$$

5: For each period t during the k -th episode, set

$$p_t \leftarrow g(\hat{\theta}^k \cdot x_t) \quad (10)$$

Algorithm 1: RMLP policy for dynamic pricing

We estimate θ^* using a regularized maximum-likelihood estimator; see Eq. (8) where the (normalized) negative log-likelihood function for θ is given by Eq. (9). We note that as a consequence of the log concavity assumption on F and $1 - F$, the optimization problem (8) is convex and can be solved efficiently.

Observe that by design, prices posted in the k -th episode are independent from the market value noises in this period, i.e., $\{z_t\}_{t=\tau_k}^{\tau_{k+1}-1}$. This allows us to estimate θ^* for each episode separately; see Theorem 2 in Section 4.1.

The lengths of episodes in our algorithm increase geometrically ($\tau_k = 2^{k-1}$), allowing for more accurate estimate of θ^* as the episode index grows. The algorithm terminates at the end of the horizon (period T), but note that it does not need to know the length of horizon in advance.

Regularization parameter λ_k constrains the ℓ_1 norm of the estimator $\hat{\theta}^k$. Selecting the value of λ_k is of crucial importance as it effects the estimator error. We set it as $\lambda_k =$

$O\left(\sqrt{\log(\tau_{k-1}d)/\tau_{k-1}}\right)$. More precisely, define

$$\begin{aligned} M &\equiv W + \varphi^{-1}(0), \\ u_M &\equiv \sup_{|x| \leq M} \left\{ \max \left\{ \log' F(x), -\log'(1 - F(x)) \right\} \right\}, \end{aligned} \quad (11)$$

where the derivatives are w.r.t. x . We note that M is an upper-bound on the maximum price offered and also, by the log-concavity property of F and $1 - F$, we have $u_M = \max \left\{ \log' F(-M), -\log'(1 - F(M)) \right\}$. Hence, u_M captures the steepness of $\log F$.

In order to minimize the regret, we run the RMLP policy with

$$\lambda_k = 4u_M \sqrt{\frac{\log(\tau_{k-1}d)}{\tau_{k-1}}}. \quad (12)$$

Note that exploration and exploitation tasks are mixed in our algorithm. In the beginning of each episode, we use what is learned from previous episode to improve the estimation of θ^* and then we exploit this estimate throughout the current episode to incur little regret. Meanwhile, the observations gathered in the current episode are used to update our estimate of θ^* for the next episode. We analyze the performance of RMLP in the next section.

4. Regret analysis

Although the description of RMLP is oblivious to sparsity s_0 , its performance depends on the structure of the optimal solution. The following theorem bounds the regret of our dynamics pricing policy.

THEOREM 1 (Regret). *Suppose Assumptions 1 and 2 hold. Then, the regret of the RMLP policy is of $O(s_0 \log T (\log d + \log T))$.*

Here we provide an outline for the proof of Theorem 1:

1. First, we bound the ℓ_2 norm estimation error for the regularized log-likelihood estimate in the k -th episode, $\hat{\theta}^k$. More specifically, we show that

$$\|\hat{\theta}^k - \theta^*\|_2 = O(\sqrt{s_0} \lambda_k) = O\left(\sqrt{\frac{s_0 \log(\tau_{k-1}d)}{\tau_{k-1}}}\right).$$

As expected, the estimate gets more accurate as the episode's length increases.

2. Let R_t be the regret occurred at step t . Also, let $h_t(p) = p(1 - F(p - x_t \cdot \theta^*))$ be the expected revenue at time t if price p_t is posted, where the expectation is w.r.t. the market noise z_t . We bound R_t in terms of $h_t(p_t^*) - h_t(p_t)$. Since $p_t^* \in \arg \max\{h_t(p)\}$, we have $h'_t(p_t^*) = 0$, and by Taylor expansion of h_t around p_t^* , we obtain $h_t(p_t^*) - h_t(p_t) = O((p_t^* - p_t)^2)$.

3. For t in the k -th episode, namely $\tau_{k-1} \leq t \leq \tau_k - 1$, we have

$$p_t^* - p_t = g(\theta^* \cdot x_t) - g(\hat{\theta}^k \cdot x_t) \leq |(\theta^* - \hat{\theta}^k) \cdot x_t|,$$

which follows by showing that g is 1-Lipschitz. Applying Assumption 2, we have $\mathbb{E}[|(\theta^* - \hat{\theta}^k) \cdot x_t|^2] \leq C_{\max} \|\theta^* - \hat{\theta}^k\|_2^2$. Combining above bounds, we arrive at $\mathbb{E}[R_t] = O(s_0 \log(\tau_{k-1}d)/\tau_{k-1})$. Therefore, the cumulative expected regret in episode k works out at $O(s_0 \log(Td))$. Since the length of episodes increase geometrically, there are $O(\log T)$ episodes by time T . This implies that the total expected regret by time T is $O(s_0 \log(d) \log(T) + s_0 [\log T]^2)$.

In the sequel, we elaborate on the above steps.

4.1. Estimating θ^*

Following step 1 of the proof outline mentioned above, we consider the problem of estimating θ^* based on observations $\{(x_1, p_1, y_1), \dots, (x_n, p_n, y_n)\}$. We make the following assumption which is satisfied by the RMLP policy, at each episode.

ASSUMPTION 3. *Sequence of posted prices p_1, p_2, \dots, p_n are independent of the market noise values z_1, z_2, \dots, z_n .*

Using probabilistic model (3), θ^* is obtained by solving a regularized maximum likelihood (ML) optimization problem. The (normalized) negative log-likelihood function for θ is as follows

$$\mathcal{L}(\theta) = -\frac{1}{n} \sum_{t=1}^n \left\{ \mathbb{I}(y_t = 1) \log(1 - F(p_t - \theta \cdot x_t)) + \mathbb{I}(y_t = -1) \log(F(p_t - \theta \cdot x_t)) \right\}. \quad (13)$$

Parameter θ is estimated as the solution of the following program:

$$\hat{\theta} = \arg \min_{\|\theta\|_1 \leq W} \mathcal{L}(\theta) + \lambda \|\theta\|_1 \quad (14)$$

For M given by (11), we define ℓ_M which corresponds to the “flatness” of function $\log F$.

$$\ell_M \equiv \inf_{|x| \leq M} \left\{ \min \left\{ -\log'' F(x), -\log''(1 - F(x)) \right\} \right\}. \quad (15)$$

By Assumption 1, the log-concavity property of F and $1 - F$, we have $\ell_M > 0$.

The next theorem upper bounds the estimation error of the proposed regularized estimator.

THEOREM 2 (Estimation Errors). *Consider linear model (1) under Assumptions 1 and 2. Also, suppose that $\theta^* \in \Omega$ and the posted prices p_1, p_2, \dots, p_n satisfy Assumption 3. Let $M \geq W$ be an upper bound on the prices and $\hat{\theta}$ be the solution of optimization problem (21) with $\lambda \geq 4u_M \sqrt{\frac{\log(nd)}{n}}$. Then, there exist positive constants c_0 and C such that, for $n \geq c_0 s_0 \log(d)$, the following inequality holds with probability at least $1 - 1/(dn^2) - 2e^{-n/(c_0 s_0)}$:*

$$\|\hat{\theta} - \theta^*\|_2^2 \leq C s_0 \lambda^2 / \ell_M^2. \quad (16)$$

We refer to Section E for the proof of Theorem 2.

As we see the ℓ_2 estimation error scales linearly with the sparsity level s_0 . As s_0 increases, the number of parameters to be estimated becomes larger and this makes the estimation problem harder, leading to worse ℓ_2 bound for a fixed number of samples, n . Further, choosing $\lambda \sim \sqrt{\log(nd)/n}$ (where \sim indicates equality up to a constant factor), our ℓ_2 bound scales logarithmically in the dimension of the demand space, d . This allows to deal with high-dimensional applications and obtain a regret that scales logarithmically in d . Further, the estimation error shrinks as $\sim 1/n$; getting more samples with fixed value of s_0, d leads to better estimation accuracy. Finally, note that for small values of ℓ_M , the log-likelihood function is very flat and there can be, in principle, vectors θ of log-likelihood value very close to the optimum and nevertheless far from the optimum. In other words, estimation task becomes harder as ℓ_M gets smaller and this is clearly reflected in the derived estimation bound.

5. Lower bound on regret

As discussed in Section 2.2, if the true parameter θ^* is known, the optimal policy (in terms of expected revenue) is the one that sets prices as $p_t = g(x_t \cdot \theta^*)$. Let $\bar{\mathcal{H}}_t = \{x_1, x_2, \dots, x_t, z_1, z_2, \dots, z_t\}$ denote the history set up to time t , and recall that Ω denotes the set of feasible θ^* , i.e., $\Omega = \{\theta \in \mathbb{R}^d : \|\theta\|_0 \leq s_0\} \cap \{\theta \in \mathbb{R}^d : \|\theta\|_1 \leq W\}$. We consider the following set of policies, Π , as follows:

$$\Pi = \left\{ \pi : \pi(p_t) = g(x_t \cdot \theta_t), \text{ for some } \theta_t \in \Omega, \text{ such that } \theta_t \text{ is } \mathcal{H}_{t-1}\text{-measurable} \right\}. \quad (17)$$

Here $\pi(p_t)$ denotes the price posted by policy π at time t .

We provide a lower bound on the achievable regret by any policy in set Π . Indeed this lower bound applies to an oracle who fully observes the market values after the price is either accepted or rejected. Compared to our setting, where the seller observes only the binary feedbacks (purchase/no purchase), this oracle appears exceedingly powerful at first sight but surprisingly, the derived lower bound matches the regret of our dynamic policy, up to a logarithmic factor.

THEOREM 3. *Consider linear model (1) where the market values $v(x_t)$, $1 \leq t \leq T$, are fully observed. We further assume that market value noises are generated as $z_t \sim \mathcal{N}(0, \sigma^2)$. Let Π be the set of policies given by (17). Then, there exists constant $C' > 0$ (depending on W and σ), such that the following holds true for all $T \in \mathbb{N}$.*

$$\min_{\pi \in \Pi} \text{Regret}_\pi(T) \geq C' \left\{ s_0 \log \left(\frac{T}{s_0} \right) + \min \left[\frac{T}{s_0}, s_0 \log \left(\frac{d}{s_0} \right) \right] \right\}. \quad (18)$$

In the following we give an outline for the proof of Theorem 3, summarizing its main steps and defer the complete proof to Section C.

1. We derive a lower bound for regret in terms of the minimax estimation error. Specifically, we show that $\max_{\theta^* \in \Omega} \mathbb{E}(R_t) \geq c \max_{\theta^* \in \Omega} \mathbb{E}\{\min(\|\theta_t - \theta^*\|_2^2, C)\}$, for some constants $c, C > 0$.

2. Let $\theta_1^T = (\theta_t)_{t=1}^T$ and define $d(\theta_1^T, \theta) \equiv \sum_{t=1}^T \min(\|\theta_t - \theta\|_2^2, C)$. We use a standard argument that relates the minimax ℓ_2 risk, $\min_{\theta_1^T} \max_{\theta^* \in \Omega} \mathbb{E}d(\theta_1^T, \theta^*)$, in terms of the error in multi-way hypothesis problem. We first construct a maximal set of points in Ω , such that minimum pairwise distances among them is at least δ . (Such set is usually referred to as a δ -packing in the literature). Here δ is a free parameter to be determined in the proof. We then use a standard reduction to show that any estimator with small minimax risk should necessarily solve a hypothesis testing problem over the packing set, with small error probability. More specifically, suppose that nature chooses one point from the packing set uniformly at random and conditional on nature's choice of the parameter vector, say θ^* , the market value are generated according to $\langle x_t, \theta^* \rangle + z_t$ with $z_t \sim \mathcal{N}(0, \sigma^2)$. The problem is reduced to lower bounding the error probability in distinguishing θ^* among the candidates in the packing set using the observed market values.

3. We apply Fano's inequality from information theory to lower bound the probability of error. The Fano bound involves the logarithm of the cardinality of the δ -packing set as well as the mutual information between the observations (market values) and the random parameter vector θ^* chosen uniformly at random from the packing set.

6. Nonlinear valuation function

Thus far, we have assumed that the market value follows a linear model given by $v(x_t) = \theta \cdot x_t + z_t$. In this section, we extend our results to nonlinear market value models that incorporate correlations among features and nonlinear dependencies on the features. We start by two applications where nonlinear valuation models have already been popular:

Semi-log model. In this model, $\ln v(x_t) = \theta \cdot x_t$. Semi-log model is common in hedonic pricing models and real state evaluations.

Log-log model. In this model, $\ln v(x_t) = \sum_i \theta_i \ln(x_{t,i})$. This model is also common in hedonic pricing models and real state evaluations.

Logistic model. Here, $\ln \left(\frac{v(x_t)}{1 - v(x_t)} \right) = \theta \cdot x_t$. Logistic model is a popular alternative to linear model and is widely used for instance in internet advertisements.

We extend our linear formulation to the nonlinear setting through the use of a nonlinear feature map $\phi: \mathbb{R}^d \mapsto \mathbb{R}^d$. Starting by a set of basic features x_t , we construct new features $\phi(x_t)$. An example of such map is $\phi(x) = (x_1^2, x_2, x_2x_3, \dots, x_d^4x_3)$ that captures higher order correlations among features. We then consider the model

$$v(x_t) = \psi(\theta \cdot \phi(x_t) + z_t), \quad (19)$$

for some function ψ and feature-map $\phi: \mathbb{R}^d \mapsto \mathbb{R}^d$, with $\theta \in \mathbb{R}^d$ being the parameter vector and z_t denoting the market noise. Note that the above cases (log-log model, semi-log model and logistic model) can be expressed as examples of this formulation. Specifically, $\psi(x) = e^x$, $\tilde{x}_t = x_t$ for semi-log model, $\psi(x) = e^x$, $\tilde{x}_t = \ln(x_t)$ for log-log model and $\psi(x) = e^x / (1 + e^x)$, $\tilde{x}_t = x_t$ for logistic model.

We proceed by making some mild assumptions on feature-map ϕ . For feature map $\phi = (\phi_1, \dots, \phi_d): \mathbb{R}^d \mapsto \mathbb{R}^d$, let $\mathcal{D}_\phi = \left(\frac{\partial \phi_i}{\partial x_j} \right)_{1 \leq i \leq j \leq d}$ denote its derivative. Recall that the feature vectors are sampled independently from an (unknown) distribution \mathbb{P}_X .

ASSUMPTION 4. *Suppose that the feature-map ϕ has continuous derivative. Denote by $\Sigma_\phi \equiv \mathbb{E}(\phi(x) \cdot \phi(x)^\top)$, the covariance of feature vector $\phi(x)$ under \mathbb{P}_X . We assume that there*

exist constants C_{\min} and C_{\max} such that for every eigenvalue σ of Σ_ϕ , we have $0 < C_{\min} \leq \sigma < C_{\max} < \infty$.

Invoking Assumption 1, \mathbb{P}_X has a bounded support \mathcal{X} and since ϕ has continuous derivative, it is Lipschitz on \mathcal{X} and hence the image of \mathcal{X} under ϕ remains bounded. Therefore, the new features $\phi(x_t)$ are also sampled independently from a bounded set. The condition on Σ_ϕ is analogous to that on Σ , as required by Assumption 2 for the linear setting.

Based on feature-map ϕ , validity of Assumption 4 may depend on all moments of distribution \mathbb{P}_X . We provide an alternative to this assumption, which only depends on feature-map ϕ and the second moment of \mathbb{P}_X .

ASSUMPTION 5. *Suppose that the feature-map ϕ has continuous derivative and its derivative $\mathcal{D}_\phi(x)$ is full-rank for almost all x . In addition, there exist constants C_{\min} and C_{\max} such that for every eigenvalue σ of covariance Σ , we have $0 < C_{\min} \leq \sigma < C_{\max} < \infty$.*

Let $\lambda(v) = f(v)/(1 - F(v))$ be the hazard rate function for distribution F . As we showed in the proof of Lemma E.1, if $1 - F$ is log-concave then its hazard rate is increasing. For a log-concave function ψ , we define

$$g_\psi^{-1}(v) \equiv v - \lambda^{-1}\left(\frac{\psi'(v)}{\psi(v)}\right). \quad (20)$$

Note that g_ψ^{-1} is well-defined because ψ is log-concave and λ is increasing. Therefore, the right-hand side of (20) is strictly increasing and thus invertible. Further, $(g_\psi^{-1})'(v) \geq 1$, for all v . This implies that $0 < g'_\psi(v) \leq 1$, for all v . A simple algebraic calculation shows that for $\psi(v) = v$, we have $g_\psi = g$, where g is defined by (5).

For the nonlinear model (19) we run (modified) RMLP policy as described in Algorithm 2. This is very similar to Algorithm 1, with a few slight modifications. Firstly, the features x_t are replaced by $\tilde{x}_t \equiv \phi(x_t)$. Secondly, in the regularized estimator, prices p_t are replaced by $\psi^{-1}(p_t)$. Thirdly, in the last step of algorithm prices are set as $\psi(g_\psi(\hat{\theta}^k \cdot \tilde{x}_t))$, with g_ψ defined by Equation (20).

THEOREM 4. *Let ψ be log-concave and strictly increasing. Suppose that Assumptions 1 and 4 (or its alternative, Assumption 5) hold. Then, regret of the RMLP policy described as Algorithm 2 is of $O(s_0 \log T(\log d + \log T))$.*

Input: (at time 0) function g , regularizations λ_k , W (bound on $\|\theta^*\|_1$),

Input: (arrives over time) covariate vectors $\{\tilde{x}_t \equiv \phi(x_t)\}_{t \in \mathbb{N}}$

Output: prices $\{p_t\}_{t \in \mathbb{N}}$

1: $\tau_1 \leftarrow 1$, $p_1 \leftarrow 0$, $\hat{\theta}^1 \leftarrow 0$

2: **for** each episode $k = 2, 3, \dots$ **do**

3: Set the length of k -th episode: $\tau_k \leftarrow 2^{k-1}$.

4: Update the model parameter estimate $\hat{\theta}^k$ using the regularized ML estimator obtained from observations in the previous episode:

$$\hat{\theta}^k = \arg \min_{\|\theta\|_1 \leq W} \{\mathcal{L}(\theta) + \lambda_k \|\theta\|_1\} \quad (21)$$

where $\mathcal{L}(\theta)$ is given by:

$$\mathcal{L}(\theta) = -\frac{1}{\tau_{k-1}} \sum_{t=\tau_{k-1}}^{\tau_k-1} \left\{ \mathbb{I}(y_t = 1) \log(1 - F(\psi^{-1}(p_t) - \theta \cdot \tilde{x}_t)) \right. \\ \left. + \mathbb{I}(y_t = -1) \log(F(\psi^{-1}(p_t) - \theta \cdot \tilde{x}_t)) \right\} \quad (22)$$

5: For each period t during the k -th episode, set

$$p_t \leftarrow \psi(g_\psi(\hat{\theta}^k \cdot \tilde{x}_t)) \quad (23)$$

Algorithm 2: RMLP Policy for dynamic pricing under the nonlinear setting

Note that for the examples discussed above (semi-log, log-log, linear and logistic models), ψ is log-concave and continuously differentiable.

Proof of Theorem 4 is given in Appendix D. Here, we summarize its key ingredients.

1. By increasing property of ψ , a sale occurs at period t when $z_t \geq \psi^{-1}(p_t) - \theta \cdot \tilde{x}_t$. Hence, the log-likelihood estimator for this setting reads as (22). By virtue of Assumption 4 (or its alternative, Assumption 5) we get a similar estimation error for the regularized estimator to the one in Theorem 2.

2. Similar to our derivation for linear setting, we show that the optimal pricing policy that knows θ^* in advance is given by $p_t^* = \psi(g_\psi(\theta^* \cdot \tilde{x}_t))$, where g_ψ is defined based on Equation (20).

3. The difference between the posted price and the optimal price can be bounded as $p_t - p_t^* = \psi(g_\psi(\hat{\theta}^k \cdot \tilde{x}_t)) - \psi(g_\psi(\theta^* \cdot \tilde{x}_t)) \leq L|\tilde{x}_t \cdot (\hat{\theta}^k - \theta^*)|$, for a constant $L > 0$. This bound is similar to the corresponding bound for the linear setting, and following the same lines of our regret analysis for that case, we get $R(T) = O(s_0 \log T(\log d + \log T))$.

7. Conclusion

In this work, we leverage tools from statistical learning to design a dynamic pricing policy for a setting wherein the products are described via high-dimensional features. Our policy is computationally efficient and by exploiting the structure of demand parameters, it obtains a regret that scales gracefully with the features dimension and the time horizon. Namely, the regret of our algorithm scales linearly with the sparsity of the optimal solution and logarithmically with the dimension. We also show an $O(\log^2 T)$ dependence of the regret on the length of the horizon. On the flip side, we provide a lower-bound of $O(\log T)$ on the regret of any algorithm that does not know the true parameters of the model in advance. A natural next step is providing a tight bound on the regret, closing the gap between the derived upper and lower bounds. Another step would be assuming that θ^* is not exactly sparse, but it can be well approximated by a sparse vector, i.e., $\|\theta^* - \theta_{s_0}\|_1 \leq \delta$ for some s_0 -sparse vector θ_{s_0} . The question is how the regret of RMLP scales with δ . Another direction for future research include extensions to time-varying environments where the parameters of the market are changing over time and to models with more general valuation functions. We also believe the ideas and techniques developed in this work can be applied to other settings such as personalized pricing where information about the buyers can be used for price differentiation or optimizing reserve prices in online ad auctions. Another application would be assortment optimization and learning consumer choice models both in terms of the role of the structure (Farias et al. 2013, Kallus and Udell 2016) as well as personalization (Golrezaei et al. 2014, Chen et al. 2015) in data-rich environments.

Appendix A: Proof of Theorem 1

We start by establishing some useful properties of the virtual valuation function φ and the price function g .

LEMMA 1. *If $1 - F$ is log-concave, then the virtual valuation function φ is strictly monotone increasing.*

LEMMA 2. *If $1 - F$ is log-concave, then the price function g satisfies $0 < g'(v) < 1$, for all values of $v \in \mathbb{R}$.*

Proofs of Lemma 1 and 2 are given in Appendix E.1 and E.2, respectively.

Given that $\|\hat{\theta}^k\|_1 \leq W$ and $|x_t \cdot \hat{\theta}^k| \leq W$ for all t, k , by applying Lemma 2 we get

$$p_t = g(x_t \cdot \theta^k) \leq g(0) + |x_t \cdot \theta^k| \leq \varphi^{-1}(0) + W. \quad (24)$$

Further $\varphi(0) < 0$ which implies $\varphi^{-1}(0) > 0$ by monotonicity of φ , as per Lemma 1. Letting $M = \varphi^{-1}(0) + W$, we have that $M > W$ serves an upper bound on the prices posted by the RLM algorithm, as required by Theorem 2.

We are now ready to bound the regret of our policy. For $t \in \mathbb{N}$, let

$$R_t \equiv p_t^* \mathbb{I}(v_t \geq p_t^*) - p_t \mathbb{I}(v_t \geq p_t) \quad (25)$$

be the regret occurred at step t . Further, let $\mathcal{H}_t = \{x_1, x_2, \dots, x_t, x_{t+1}, z_1, z_2, \dots, z_t\}$ be the history set up to time t , augmented by the new feature x_{t+1} .

We write

$$\mathbb{E}(R_t | \mathcal{H}_{t-1}) = \mathbb{E}(p_t^* \mathbb{I}(v_t \geq p_t^*) | \mathcal{F}_{t-1}) - \mathbb{E}(p_t \mathbb{I}(v_t \geq p_t) | \mathcal{F}_{t-1}) \quad (26)$$

$$= p_t^* (1 - F(p_t^* - x_t \cdot \theta^*)) - p_t (1 - F(p_t - x_t \cdot \theta^*)) \quad (27)$$

Define function $h_t : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ as $h_t(p) \equiv p(1 - F(p - x_t \cdot \theta^*))$. Note that $p_t^* \in \arg \max h_t(p)$ and thus $h'(p_t^*) = 0$. By Taylor expansion,

$$h_t(p_t) = h_t(p_t^*) + \frac{1}{2} h_t''(p)(p_t - p_t^*)^2, \quad (28)$$

for some p between p_t and p_t^* . Further

$$|h_t''(p)| = |1 - F(p - x_t \cdot \theta^*) - pf(p - x_t \cdot \theta^*)| \leq 1 + pf(p - x_t \cdot \theta^*) \leq 1 + M \cdot \max_z f(z), \quad (29)$$

where we use the fact that $p_t, p_t^* \leq M$ and consequently $p \leq M$. We denote the right hand side of (29) by C_1 .

Combining Equations (27), (28), (29), along with 1-Lipschitz property of g gives

$$\mathbb{E}(R_t | \mathcal{H}_{t-1}) \leq \frac{C_1}{2} (p_t^* - p_t)^2 = \frac{C_1}{2} (g(\theta^* \cdot x_t) - g(\hat{\theta}^k \cdot x_t))^2 \leq \frac{C_1}{2} |x_t \cdot (\theta^* - \hat{\theta}^k)|^2. \quad (30)$$

We use the shorthand $\Delta_t = \max \left\{ \mathbb{E}(R_t) : \|\theta^*\|_1 \leq R, \mathbb{P}_X \in \mathcal{Q}(\mathcal{X}) \right\}$. In the sequel, we bound Δ_t by considering two cases. The first case is $t \leq t_0$ with $t_0 = c_0 s_0 \log(d)$ and c_0 the constant in the statement of Theorem 2. In this case, episodes are not large enough to estimate θ^* accurately enough, and thus we provide a uniform bound on Δ_t . The second case is $t > t_0$, where the estimate of θ is accurate enough due to Theorem 2 and we give finer bound on Δ_t .

- *Case $t \leq t_0$.* Clearly, by (27) we have $\mathbb{E}(R_t | \mathcal{H}_{t-1}) \leq p_t^* \leq M$. Hence, $\Delta_t \leq M$ in this case.
- *Case $t > t_0$.* Fix t and assume that it is within the k^{th} episode, $\tau_k \leq t \leq \tau_{k+1} - 1$. We let $\bar{\mathcal{H}}_t = \mathcal{H}_t \setminus \{x_{t+1}\}$. Then by Equation (30), $\mathbb{E}(R_t | \bar{\mathcal{H}}_{t-1}) \leq (C_1 C_{\max}/2) \|\hat{\theta}^k - \theta^*\|_2^2$. Define the probability event $\mathcal{G}_k \equiv \left\{ \|\hat{\theta}^k - \theta^*\|_2^2 \leq C s_0 \lambda^2 / \ell_M^2 \right\}$, with C the same constant in (16) and proceed as follows

$$\begin{aligned} \mathbb{E}(R_t) &= \mathbb{E}(\mathbb{E}(R_t | \bar{\mathcal{H}}_{t-1})) = \mathbb{E}(\mathbb{E}(R_t | \bar{\mathcal{H}}_{t-1}; \mathcal{G}_k) + \mathbb{E}(\mathbb{E}(R_t | \bar{\mathcal{H}}_{t-1}; \mathcal{G}_k^c)) \\ &\leq C_1 C_{\max} C \frac{s_0 \lambda^2}{2 \ell_M^2} + M \mathbb{P}(\mathcal{G}_k^c). \end{aligned} \quad (31)$$

Invoking Theorem 2,

$$\mathbb{P}(\mathcal{G}_k^c) \leq \frac{1}{d \tau_{k-1}^2} + 2e^{-\frac{\tau_{k-1}}{c_0 s_0}}. \quad (32)$$

We are now ready to bound regret of our dynamic pricing policy.

Let $k_0 = \lfloor \log t_0 \rfloor + 1$ and $k_1 = \lfloor \log T \rfloor + 1$ be the indices of episodes containing t_0 and T , respectively. Combing bounds obtained on $\Delta_t^{(1)}$ and $\Delta_t^{(2)}$, we get

$$\begin{aligned} \text{Regret}(T) &\leq \sum_{t=1}^T \Delta_t \leq Mt_0 + \sum_{k=k_0}^{k_1} C_1 C_{\max} C \frac{s_0 \lambda_k^2}{2\ell_M^2} \tau_k + \sum_{k=k_0}^{k_1} M \left[\frac{1}{d \tau_{k-1}} + 2e^{-\frac{\tau_{k-1}}{c_0 s_0}} \right] \tau_k \\ &\leq Mt_0 + \sum_{k=1}^{k_1} 16C_1 C_{\max} C \frac{u_M^2}{\ell_M^2} s_0 \log(\tau_{k-1} d) + \sum_{k=k_0}^{k_1} M \left[\frac{2}{d \tau_{k-1}} + 2\tau_k e^{-\frac{\tau_{k-1}}{c_0 s_0}} \right] \\ &\leq Mt_0 + 16C_1 C_{\max} C \frac{u_M^2}{\ell_M^2} s_0 \log(Td) \log(T) + \frac{M}{d} \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} + 4c_0 M s_0 \int_{(\log d)/4}^{\infty} x e^{-x} dx \end{aligned} \quad (33)$$

where in the final step, we used the fact that $\tau_{k_0-1} \geq c_0 s_0 (\log d)/4$. Simplifying further, we get

$$\text{Regret}(T) \leq c_0 M s_0 \log d + 16C_1 C_{\max} C \frac{u_M^2}{\ell_M^2} s_0 \log T (\log d + \log T) + \frac{2M}{d} + \frac{4c_0 s_0 M}{d^{1/4}}. \quad (34)$$

Therefore $\text{Regret}(T) = O(s_0 \log T (\log d + \log T))$.

Appendix B: Proof of Theorem 2

We start by reviewing the notion of *restricted eigenvalue* (RE) which is commonplace in high-dimensional statistical estimation.

DEFINITION 1. For a given matrix $A \in \mathbb{R}^{d \times d}$ and some integer s such that $1 \leq s \leq d$ and a positive number c , we say that *Restricted Eigenvalue* (RE) condition is met if

$$\kappa(s, c)^2 \equiv \min_{\substack{J \subseteq [p] \\ |J| \leq s}} \min_{\substack{v \neq 0 \\ \|v_{J^c}\|_1 \leq c \|v_J\|_1}} \frac{v^\top A v}{\|v_J\|_2^2} > 0.$$

It is shown in (Bühlmann and van de Geer 2011) and (Rudelson and Zhou 2013) that when two matrices A_0 , A_1 are close to each other (in the maximum element-wise norm) compared to sparsity s , the RE condition for A_0 implies the RE condition for A_1 . This is particularly useful when A_0 is a population covariance matrix and A_1 is a corresponding empirical covariance matrix. More concretely, the following proposition holds true.

PROPOSITION 1. Let $\hat{\Sigma} = X^\top X/n$ be the empirical covariance matrix and $S = \text{supp}(\theta^*)$ be the support of θ^* . Under Assumption 2, $\hat{\Sigma}$ satisfies the restricted eigenvalue condition with constant $\kappa(s_0, 3) \geq \sqrt{C_{\min}/2}$, with probability $1 - e^{-2n/(c_0 s_0)}$ and $c_0 = 768/C_{\min}^2$, provided that $n \geq c_0 s_0 \log d$,

Proposition 1 follows from the results established in Bühlmann and van de Geer (2011) and Rudelson and Zhou (2013). We outline the main steps of its proof in Appendix B.1 for the reader's convenience.

By the second-order Taylor's theorem, expanding around θ^* we have

$$\mathcal{L}(\theta^*) - \mathcal{L}(\hat{\theta}) = -\langle \nabla \mathcal{L}(\theta^*), \hat{\theta} - \theta^* \rangle - \frac{1}{2} \langle \hat{\theta} - \theta^*, \nabla^2 \mathcal{L}(\tilde{\theta})(\hat{\theta} - \theta^*) \rangle, \quad (35)$$

for some $\tilde{\theta}$ on the line segment between θ^* and $\hat{\theta}$. Invoking (13), we have

$$\nabla \mathcal{L}(\theta) = \frac{1}{n} \sum_{t=1}^n \mu_t(\theta) x_t, \quad \nabla^2 \mathcal{L}(\theta) = \frac{1}{n} \sum_{t=1}^n \eta_t(\theta) x_t x_t^\top, \quad (36)$$

where ∇ and ∇^2 represents the gradient and the hessian w.r.t θ . Further,

$$\begin{aligned}\mu_t(\theta) &= -\frac{f(u_t(\theta))}{F(u_t(\theta))}\mathbb{I}(y_t = -1) + \frac{f(u_t(\theta))}{1-F(u_t(\theta))}\mathbb{I}(y_t = +1) \\ &= -\log' F(u_t(\theta))\mathbb{I}(y_t = -1) - \log'(1-F(u_t(\theta)))\mathbb{I}(y_t = +1) \\ \eta_t(\theta) &= \left(\frac{f(u_t(\theta))^2}{F(u_t(\theta))^2} - \frac{f'(u_t(\theta))}{F(u_t(\theta))}\right)\mathbb{I}(y_t = -1) + \left(\frac{f(u_t(\theta))^2}{(1-F(u_t(\theta)))^2} + \frac{f'(u_t(\theta))}{1-F(u_t(\theta))}\right)\mathbb{I}(y_t = +1) \\ &= -\log'' F(u_t(\theta))\mathbb{I}(y_t = -1) - \log''(1-F(u_t(\theta)))\mathbb{I}(y_t = +1),\end{aligned}$$

where $u_t(\theta) = p_t - \langle x_t, \theta \rangle$, and $\log' F(x)$ and $\log'' F(x)$ represent first and second derivative w.r.t x , respectively.

By our assumption, $|u_t(\theta^*)| \leq \max(P, \|x_t\|_\infty \|\theta^*\|_2) \leq \max(M, W) = M$. Further, recall that the sequences $\{p_t\}_{t=1}^n$ and $\{x_t\}_{t=1}^n$ are independent of $\{z_t\}_{t=1}^n$. Therefore, $\{u_t(\theta^*)\}_{t=1}^T$ and $\{z_t(\theta^*)\}_{t=1}^T$ are independent and by (3), we have $\mathbb{E}(\mu_t(\theta^*)) = \mathbb{E}\{\mathbb{E}(\mu_t(\theta^*)|u_t(\theta^*))\} = 0$. Further, by definition of u_M , cf. Equation (12), we have $|\mu_t(\theta^*)| \leq u_M$.

We next introduce the set

$$\mathcal{F} \equiv \left\{ \|\nabla \mathcal{L}(\theta^*)\|_\infty \leq 2u_M \sqrt{\frac{\log(nd)}{n}} \right\}.$$

By applying Azuma-Hoeffding inequality followed by union bounding over d coordinates of feature vectors, we obtain $\mathbb{P}(\mathcal{F}) \geq 1 - 1/(dn^2)$.

On the other note, $\|\theta^*\|_1, \|\hat{\theta}\|_1 \leq W$ and hence $\|\tilde{\theta}\|_1 \leq W$. This implies that $|u_t(\tilde{\theta})| \leq M$. Therefore, by definition of ℓ_M , cf. Equation (15), we have $\eta_t(\tilde{\theta}) \geq \ell_M$. Recalling Equation (36), we get $\nabla \mathcal{L}(\tilde{\theta}) \succeq \ell_M(X^\top X/n)$.

By optimality of $\hat{\theta}$, we write

$$\mathcal{L}(\hat{\theta}) + \lambda \|\hat{\theta}\|_1 \leq \mathcal{L}(\theta^*) + \lambda \|\theta^*\|_1, \quad (37)$$

and by rearranging the terms and using (35), we arrive at

$$\frac{\ell_M}{n} \|X(\theta^* - \hat{\theta})\|^2 + \lambda \|\hat{\theta}\|_1 \leq \|\nabla \mathcal{L}(\theta^*)\|_\infty \|\hat{\theta} - \theta^*\|_1 + \lambda \|\theta^*\|_1. \quad (38)$$

Choosing $\lambda \geq 4u_M \sqrt{(\log(nd))/n}$, we have on \mathcal{F}

$$\frac{2\ell_M}{n} \|X(\theta^* - \hat{\theta})\|^2 + 2\lambda \|\hat{\theta}\|_1 \leq \lambda \|\hat{\theta} - \theta^*\|_1 + 2\lambda \|\theta^*\|_1. \quad (39)$$

Let $S = \text{supp}(\theta^*)$. On the left-hand side using triangle inequality, we have

$$\|\hat{\theta}\|_1 = \|\hat{\theta}_S\|_1 + \|\hat{\theta}_{S^c}\|_1 \geq \|\hat{\theta}_S\|_1 - \|\hat{\theta}_S - \theta_S^*\|_1 + \|\hat{\theta}_{S^c}\|_1.$$

On the right-hand side, we have

$$\|\hat{\theta} - \theta^*\|_1 = \|\hat{\theta}_S - \theta_S^*\|_1 + \|\hat{\theta}_{S^c}\|_1.$$

Using these two inequalities in (39), we get

$$\frac{2\ell_M}{n} \|X(\theta^* - \hat{\theta})\|^2 + \lambda \|\hat{\theta}_{S^c}\|_1 \leq 3\lambda \|\hat{\theta}_S - \theta_S^*\|_1. \quad (40)$$

We next write

$$\begin{aligned}
\frac{2\ell_M}{n} \|X(\theta^* - \hat{\theta})\|^2 + \lambda \|\hat{\theta} - \theta^*\|_1 &= \frac{2\ell_M}{n} \|X(\theta^* - \hat{\theta})\|^2 + \lambda \|\hat{\theta}_S - \theta_S^*\|_1 + \lambda \|\hat{\theta}_{S^c}\|_1 \\
&\stackrel{(a)}{\leq} 4\lambda \|\hat{\theta}_S - \theta_S^*\|_1 \stackrel{(b)}{\leq} 4\lambda \sqrt{s_0} \|\hat{\theta}_S - \theta_S^*\|_2 \\
&\stackrel{(c)}{\leq} \frac{4\lambda \sqrt{2s_0}}{\sqrt{nC_{\min}}} \|X(\hat{\theta} - \theta^*)\|_2 \\
&\stackrel{(d)}{\leq} \frac{\ell_M}{n} \|X(\hat{\theta} - \theta^*)\|_2^2 + \frac{8\lambda^2 s_0}{\ell_M C_{\min}},
\end{aligned}$$

where (a) follows from Equation (40); (b) holds by Cauchy-Schwarz inequality; (c) follows from the RE condition, which holds for $\hat{\Sigma} = (X^\top X)/n$ as stated by Proposition 1, with $\kappa(s_0, 3) \geq \sqrt{C_{\min}}$, and recalling the inequality $\|\hat{\theta}_{S^c} - \theta_{S^c}^*\|_1 = \|\hat{\theta}_{S^c}\|_1 \leq 3\|\hat{\theta}_S - \theta_S^*\|_2$ as per Equation (40); Finally (d) follows from the inequality $2\sqrt{ab} \leq a^2 + b^2$. Rearranging the terms, we obtain

$$\frac{\ell_M}{n} \|X(\theta^* - \hat{\theta})\|^2 + \lambda \|\hat{\theta} - \theta^*\|_1 \leq \frac{8\lambda^2 s_0}{\ell_M C_{\min}}. \quad (41)$$

Applying the RE condition again to the L.H.S of (41), we get

$$C_{\min} \frac{\ell_M}{2} \|\theta^* - \hat{\theta}\|_2^2 \leq \frac{\ell_M}{n} \|X(\theta^* - \hat{\theta})\|^2 \leq \frac{8\lambda^2 s_0}{\ell_M C_{\min}}, \quad (42)$$

and therefore,

$$\|\theta^* - \hat{\theta}\|_2^2 \leq \frac{16s_0\lambda^2}{\ell_M^2 C_{\min}^2}. \quad (43)$$

The result follows.

B.1. Proof of Proposition 1

We outline the key steps of the proof which are established in .

The result holds for a more general case where X has subgaussian entries.

DEFINITION 2. A random variable ν is subgaussian if there exist constants L, σ_0 such that

$$\mathbb{E}(e^{\frac{\nu^2}{L^2}}) \leq \frac{\sigma_0^2}{L^2} + 1.$$

Note that bounded random variables are subgaussian. Specifically, if $|\nu| \leq \nu_{\max}$, then ν is subgaussian with $L = \nu_{\max}$ and $\sigma_0 = \nu_{\max} \sqrt{e-1}$.

For a matrix A , we let $\|A\|_\infty$ denote its (element wise) maximum norm, i.e., $\|A\|_\infty = \max_{i,j} |A_{ij}|$. The next lemma shows that if two matrices are close enough in maximum norm and if the compatibility condition holds for one of them then it would also hold for the other one.

LEMMA 3. Suppose that the restricted eigenvalue (RE) condition holds for Σ_0 with constant $\kappa_{\Sigma_0}(s_0, 3) > 0$. If

$$\|\Sigma_0 - \Sigma_1\|_\infty \leq \frac{\kappa_{\Sigma_0}^2(s_0, 3)}{32s_0},$$

then the RE condition holds for Σ_1 with constant $\kappa_{\Sigma_1}(s_0, 3) \geq \kappa_{\Sigma_0}(s_0, 3)/\sqrt{2}$.

Proof of Lemma 3 We refer to Problem 6.10 of Bühlmann and van de Geer (2011).

LEMMA 4. Consider $X \in \mathbb{R}^{n \times p}$ with i.i.d. rows generated from a distribution with covariance $\Sigma \in \mathbb{R}^{d \times d}$. Let $\hat{\Sigma} = X^\top X/n$ be the corresponding empirical covariance. Further, suppose that the entries of X are uniformly subgaussian with parameters L, σ_0 . If $n \geq c_0 L s_0 \log d$ with $c_0 = 768L/\kappa_{\Sigma_0}^2(s_0, 3)$, then

$$\mathbb{P} \left[\|\hat{\Sigma} - \Sigma\|_\infty \geq \frac{\kappa_{\Sigma_0}^2(s_0, 3)}{384s_0} \left(2 + \frac{7\sigma_0}{L} \right) \right] \leq e^{-\frac{2n}{cs_0}}. \quad (44)$$

Proof of Lemma 4 The result follows readily from Problem 14.3 on page 535 of Bühlmann and van de Geer (2011).

Next we note that Σ satisfies the restricted eigenvalue condition with constant $\kappa_\Sigma(s_0, 3) \geq \sqrt{C_{\min}}$ because of Assumption 2. Further, since $\|x_i\|_\infty \leq 1$, we can apply the result of Lemma 4 with $L = 1$, $\sigma_0 = \sqrt{e-1}$. Proposition 1 then follows from Lemma 3.

Appendix C: Proof of Theorem 3

The regret benchmark (7) is defined as the maximum gap between a policy and the oracle policy over different $\theta^* \in \Omega$ and $p_X \in Q(\mathcal{X})$. Without loss of generality, we assume $\mathcal{X} = [-1, 1]^d$. In order to obtain a lower bound on the regret, it suffices to consider a specific distribution in $Q(\mathcal{X})$. We consider a distribution p_X that selects coordinates x_i , $1 \leq i \leq d$, uniformly at random from $\{-1, 1\}$ and independent of each other.

Fix an arbitrary policy in this family. Recalling our notation in the proof of Theorem 1, R_t denotes the regret occurred at step t and by Equations (27), (28), we have

$$\mathbb{E}(R_t | \mathcal{H}_{t-1}) = h_t(p_t^*) - h_t(p_t) = -\frac{1}{2} h_t''(p)(p_t - p_t^*)^2, \quad (45)$$

for some p between p_t and p_t^* .

We next state two lemmas which will be used in lower bounding $\mathbb{E}(R_t | \mathcal{F}_{t-1})$.

LEMMA 5. There exists a constant $c_1 > 0$ (depending on W and σ) such that, with probability one¹, $h_t''(p_t^*) \leq -c_1$. Further, there exists constant $\delta > 0$ (depending on W and σ) such that $h_t''(p) \leq -c_1/4$ for $p \in [p_t^* - \delta, p_t^* + \delta]$, with probability one.

Proof of Lemma 5 is given in Appendix E.3.

Since function h has only one local maximum, namely p_t^* , the function is increasing before p_t^* and decreasing afterward. Therefore, if $p_t \leq p_t^* - \delta$ then

$$h_t(p_t) \leq h_t(p_t^* - \delta) = h_t(p_t^*) + \frac{1}{2} h_t''(p) \delta^2 \leq h_t(p_t^*) - \frac{c_1}{8} \delta^2, \quad (46)$$

where p is some point in $[p_t^* - \delta, p_t^*]$ and we applied Lemma 5 in the last step.

Similarly, for $p_t \geq p_t^* + \delta$ we obtain

$$h_t(p_t) \leq h_t(p_t^* + \delta) = h_t(p_t^*) + \frac{1}{2} h_t''(p) \delta^2 \leq h_t(p_t^*) - \frac{c_1}{8} \delta^2, \quad (47)$$

where $p \in [p_t^* - \delta, p_t^*]$ this time. Combining these two inequalities, we get that $h_t(p_t^*) - h_t(p_t) \geq c_1 \delta^2/8$ if $|p_t^* - p_t| \geq \delta$. Hence we have

$$\mathbb{E}(R_t | \mathcal{F}_{t-1}) \geq h_t(p_t^*) - h_t(p_t) = \begin{cases} \frac{c_1}{8} (p_t - p_t^*)^2, & \text{if } |p_t - p_t^*| \leq \delta, \\ \frac{c_1}{8} \delta^2, & \text{if } |p_t - p_t^*| \geq \delta. \end{cases} \quad (48)$$

¹ The randomness comes from randomness in prices which in turn comes from randomness in features x_t .

We proceed by relating the lower bound to the error in estimation θ^* .

$$\mathbb{E}(R_t|\mathcal{F}_{t-1}) \geq \frac{c_1}{8} \min\{(p_t - p_t^*)^2, \delta^2\} = \frac{c_1}{8} \min\{(g(x_t \cdot \theta_t) - g(x_t \cdot \theta^*))^2, \delta^2\} \quad (49)$$

$$\geq \frac{c_1}{8} \min\{c_2^2 |x_t \cdot (\theta_t - \theta^*)|^2, \delta^2\}, \quad (50)$$

where we used the fact that by Lemma 2, $g'(v) > c_2$ over the bounded interval $[-W, W]$, for some constant $c_2 > 0$. Recall the definition $\bar{\mathcal{H}}_t \equiv \mathcal{H}_t \setminus \{x_{t+1}\} = \{x_1, x_2, \dots, x_t, z_1, z_2, \dots, z_t\}$. Since $\bar{\mathcal{H}}_t \subseteq \mathcal{H}_t$, by iterated law of expectation, we get

$$\mathbb{E}(R_t|\bar{\mathcal{H}}_{t-1}) = \mathbb{E}(\mathbb{E}(R_t|\mathcal{H}_{t-1})|\bar{\mathcal{H}}_{t-1}) \geq \frac{c_1}{8} \mathbb{E}(\min\{c_2^2 \|x_t \cdot (\theta_t - \theta^*)\|_2^2, \delta^2\}|\bar{\mathcal{H}}_{t-1}) \quad (51)$$

Note that x_t is independent of $\bar{\mathcal{H}}_{t-1}$ and $\theta_t - \theta^*$ is \mathcal{H}_{t-1} -measurable.

We use the following lemma to lower bound the right-hand side of (51).

LEMMA 6. *Let $x \in \mathbb{R}^d$ be a random vector such that its coordinates are chosen independently and uniformly at random from $\{-1, 1\}$. Further, suppose that $v \in \mathbb{R}^d$ and $\delta > 0$ are deterministic. Then,*

$$\mathbb{E}(\min\{(x \cdot v)^2, \delta^2\}) \geq 0.1 \min(\|v\|_2^2, \delta^2). \quad (52)$$

Proof of Lemma 6 is given in Appendix E.4.

Applying Lemma 6 to bound (51), we obtain

$$\mathbb{E}(R_t|\bar{\mathcal{H}}_{t-1}) \geq \frac{c_1 c_2^2}{80} \min\left(\|\theta_t - \theta^*\|_2^2, \frac{\delta^2}{c_2^2}\right). \quad (53)$$

Equation (51) lower bounds the expected regret at each step to the ℓ_2 estimation error.

We continue by establishing a minimax lower bound on ℓ_2 -risk of estimation.

LEMMA 7. *Consider linear model (1) where the market values $v(x_t)$, $1 \leq t \leq T$, are fully observed and the feature vectors are generated according to p_X , described above. We further assume that the noise in market value is generated as $z_t \sim \mathcal{N}(0, \sigma^2)$. For a sequence of estimators θ_t , we let $\theta_1^t = (\theta_1, \theta_2, \dots, \theta_t)$. Then, conditional on feature vectors (x_1, \dots, x_T) , and for any fixed value $C > 0$, there exist a nonnegative constant \tilde{C} , depending on C, σ, W , such that*

$$\min_{\theta_1^T} \max_{\theta^* \in \Omega} \sum_{t=1}^T \mathbb{E}(\min(\|\theta_t - \theta^*\|_2^2, C)) \geq \tilde{C} \min\left\{\frac{T}{s_0} + s_0 \log\left(\frac{T}{s_0}\right), s_0 \log\left(\frac{dT}{s_0^2}\right)\right\}. \quad (54)$$

Proof of Lemma 7 is given in Appendix E.5.

We are now ready to lower bound the regret of any policy in Π .

$$\text{Regret}(T) \geq \max_{\theta^* \in \Omega} \sum_{t=1}^T \mathbb{E}(R_t) \geq \frac{c_1 c_2^2}{80} \sum_{t=1}^T \mathbb{E}\left\{\min\left(\|\theta_t - \theta^*\|_2^2, \frac{\delta^2}{c_2^2}\right)\right\} \quad (55)$$

$$\geq \tilde{C} \frac{c_1 c_2^2}{80} \min\left\{\frac{T}{s_0} + s_0 \log\left(\frac{T}{s_0}\right), s_0 \log\left(\frac{dT}{s_0^2}\right)\right\}. \quad (56)$$

where the last step follows from Lemma 7.

Appendix D: Proof of Theorem 4

Let $\tilde{x}_t \equiv \phi(x_t)$ denote the transformed features under the feature-map, and $\tilde{p}_t = \psi^{-1}(p_t)$. We first show that Assumption 5 implies Assumption 4, and therefore it suffices to prove the theorem under Assumption 4.

LEMMA 8. *Suppose that Assumption 1 hold true. Then, Assumption 5 implies Assumption 4.*

Proof of Lemma 8 is given in Appendix D.1.

By Assumption 1, the support of \mathbb{P}_X is abounded set \mathcal{X} . Given that ϕ has a continuous derivative, it is Lipschitz on the bounded set \mathcal{X} and ergo the image of \mathcal{X} remains bounded under the feature-map ϕ . Putting differently, features \tilde{x}_t are sampled from a bounded set in \mathbb{R}^d . Without loss of generality, we assume $\|\tilde{x}_t\|_\infty \leq 1$. Further, as per Assumption 4, the covariance of the underlying distribution Σ_ϕ is positive definite with bounded eigenvalues.

On a different note, since ψ is strictly increasing, a sale occurs at period t when $\theta \cdot \tilde{x}_t + z_t \geq \psi^{-1}(p_t) = \tilde{p}_t$. Therefore the (negative) log-likelihood function for θ reads as

$$\mathcal{L}(\theta) = -\frac{1}{\tau_{k-1}} \sum_{t=\tau_{k-1}}^{\tau_k-1} \left\{ \mathbb{I}(y_t = 1) \log(1 - F(\tilde{p}_t - \theta \cdot \tilde{x}_t)) + \mathbb{I}(y_t = -1) \log(F(\tilde{p}_t - \theta \cdot \tilde{x}_t)) \right\}.$$

The estimation bound (16) also holds for this setting and the proof goes along the same lines of the proof of Theorem 2 with slight modifications: (i) the features x_t and prices p_t should be replaced by \tilde{x}_t and \tilde{p}_t . (ii) Quantity $M \geq W$ in the statement of Theorem 2 should be set as $M = g_\psi(0) + W$. This follows from the bounds below

$$\tilde{p}_t = g_\psi(\tilde{x}_t \cdot \hat{\theta}^k) \leq g_\psi(0) + |\tilde{x}_t \cdot \hat{\theta}^k| \leq g_\psi(0) + W. \quad (57)$$

Here, we used the facts that g_ψ is 1-Lipschitz and increasing as explained below Equation (20).

We next characterize the optimal policy when the true parameter θ^* is known. The expected revenue from a poster price p works out at $p(1 - F(\psi^{-1}(p) - \theta^* \cdot \tilde{x}_t))$. Writing this in terms of $\tilde{p} = \psi^{-1}(p)$, the first order condition for the optimal price reads as

$$\lambda(\tilde{p}^* - \theta^* \cdot \tilde{x}_t) \equiv \frac{f(\tilde{p}^* - \theta^* \cdot \tilde{x}_t)}{1 - F(\tilde{p}^* - \theta^* \cdot \tilde{x}_t)} = \frac{\psi'(\tilde{p}^*)}{\psi(\tilde{p}^*)}, \quad (58)$$

where λ denotes the hazard rate function. Equivalently

$$\theta^* \cdot \tilde{x}_t = \tilde{p}^* - \lambda^{-1}\left(\frac{\psi'(\tilde{p}^*)}{\psi(\tilde{p}^*)}\right). \quad (59)$$

By definition of function g_ψ as per Equation (20), we get $\tilde{p}^* = g_\psi(\theta^* \cdot \tilde{x}_t)$ and thus $p^* = \psi(g_\psi(\theta^* \cdot \tilde{x}_t))$.

We are now ready to bound the regret of the algorithm.

Let $R_t = p_t^* \mathbb{I}(v_t \geq p_t^*) - p_t \mathbb{I}(v_t \geq p_t)$ be the regret occurred at period t . Further, let $\mathcal{H}_t = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_t, \tilde{x}_{t+1}, z_1, \dots, z_t\}$ denote the history up to time t , augmented by the new feature \tilde{x}_{t+1} . Similar to the linear setting, we let $t_0 = c_0 s_0 \log(d)$ and consider two cases:

- Case $t \leq t_0$. We have $\mathbb{E}(R_t | \mathcal{H}_{t-1}) \leq p_t^* = \psi(\tilde{p}_t^*) \leq \psi(M)$ by increasing property of ψ and since $\tilde{p}_t^* \leq M$, by (57).

• Case $t > t_0$. Suppose that t belongs to the k^{th} episode, i.e., $\tau_k \leq t \leq \tau_{k+1} - 1$. Similar to Equation (30), we have

$$\begin{aligned} \mathbb{E}(R_t | \mathcal{H}_{t-1}) &\leq \frac{C_1}{2} (p_t^* - p_t)^2 = \frac{C_1}{2} \left[\psi(g_\psi(\theta^* \cdot \tilde{x}_t)) - \psi(g_\psi(\theta^{*k} \cdot \tilde{x}_t)) \right]^2 \\ &\leq \frac{C_1}{2} L(g_\psi(\theta^* \cdot \tilde{x}_t) - g_\psi(\theta^{*k} \cdot \tilde{x}_t))^2 \leq \frac{LC_1}{2} |\tilde{x}_t \cdot (\theta^* - \hat{\theta}^k)|^2, \end{aligned} \quad (60)$$

where $L \equiv \max_{|v| \leq \psi(M)} |\psi'(v)|$ (since ψ is continuously differentiable, it attains a maximum over a bounded set.) In addition, we used the fact that $g'_\psi(v) \leq 1$ as explained below Equation (20). The inequalities above then follow from the mean-value theorem.

Let $\bar{\mathcal{H}}_t = \mathcal{H}_t \setminus \{x_{t+1}\}$. Then, by Equation (60) we obtain $\mathbb{E}(R_t | \bar{\mathcal{H}}_{t-1}) \leq (LC_1 C_{\max}/2) \|\hat{\theta}^k - \theta^*\|_2^2$.

The rest of the proof goes along the same lines as in the linear setting and we obtain

$$\text{Regret}(T) \leq c_0 M s_0 \log d + 16LC_1 C_{\max} C \frac{u_M^2}{\ell_M^2} s_0 \log T (\log d + \log T) + \frac{2\psi(M)}{d} + \frac{4c_0 s_0 \psi(M)}{d^{1/4}}. \quad (61)$$

Therefore $\text{Regret}(T) = O(s_0 \log T (\log d + \log T))$.

D.1. Proof of Lemma 8

We recall the notion of *0-property* established by (Ponomarev 1987).

DEFINITION 3. A continuous function has the *0-property*, if the pre-image of any set of probability zero is a set of probability zero.

As proved in (Ponomarev 1987, Theorem 1), if a function $\phi : \mathcal{X} \subseteq \mathbb{R}^d \mapsto \mathbb{R}^d$ is continuously differentiable, then it satisfies 0-property if and only if its derivative $\mathcal{D}\phi$ is full rank for almost all $x \in \mathcal{X}$. Therefore, we need to show that under Assumption 1, if ϕ has 0-property, then Assumption 4 holds true.

Supposing otherwise, there exists a nonzero $v \in \mathbb{R}^d$ such that $v^\top \Sigma_\phi v = 0$. Therefore, $\mathbb{E}((z \cdot \phi(x))^2) = 0$ which implies that $z \cdot \phi(x) = 0$, almost surely. Define $S \equiv \{z \in \mathbb{R}^d : z \cdot \phi(x) = 0\}$. Space S is $(d-1)$ -dimensional and all the points in $\phi(\mathcal{X})$ belong to S almost surely, i.e., $\mathbb{P}(\phi(\mathcal{X}) \cap S^c) = 0$. However, since Σ is positive definite (with all of its eigenvalues target than C_{\min} , by Assumption 1), $\mathbb{P}_X(S) = 0$. Combining these observations, $\mathbb{P}(\phi(\mathcal{X})) \leq \mathbb{P}(S) + \mathbb{P}(\phi(\mathcal{X}) \cap S^c) = 0$. Since ϕ has the 0-property, this implies that $\mathbb{P}_X(\mathcal{X}) = 0$, which is a contradiction because \mathcal{X} is the support of \mathbb{P}_X and thus $\mathbb{P}_X(\mathcal{X}) = 1$. The result follows.

Appendix E: Proof of Technical Lemmas

E.1. Proof of Lemma 1

We write the virtual valuation function as $\varphi(v) = v - 1/\lambda(v)$ where $\lambda(v) = \frac{f(v)}{1-F(v)} = -\log'(1-F(v))$ is the hazard rate function. Since $1-F$ is log-concave, the hazard function $\lambda(v)$ is increasing which implies that φ is strictly increasing. Indeed, by this argument $\varphi'(v) > 1$.

E.2. Proof of Lemma 2

Recalling the definition $g(v) = v + \varphi^{-1}(-v)$, we have $g'(v) = 1 - 1/\varphi'(\varphi^{-1}(-v))$. Since φ is strictly increasing by Lemma 1, we have $g'(v) < 1$. The claim $g'(v) > 0$ follows if we show $\varphi'(\varphi^{-1}(-v)) > 1$. For this we refer to the proof of Lemma 1, where we showed that $\varphi'(v) > 1$ for all v .

E.3. Proof of Lemma 5

Let $\phi(v)$ and $\Phi(v)$ respectively denote the density and the distribution function of standard normal variable. Function h_t and its derivatives read as

$$h_t(p) = p \left(1 - \Phi \left(\frac{p - x_t \cdot \theta^*}{\sigma} \right) \right), \quad (62)$$

$$h'_t(p) = 1 - \Phi \left(\frac{p - x_t \cdot \theta^*}{\sigma} \right) - \frac{p}{\sigma} \phi \left(\frac{p - x_t \cdot \theta^*}{\sigma} \right), \quad (63)$$

$$h''_t(p) = \frac{1}{\sigma} \left[\frac{p}{\sigma} \left(\frac{p - x_t \cdot \theta^*}{\sigma} \right) - 2 \right] \phi \left(\frac{p - x_t \cdot \theta^*}{\sigma} \right). \quad (64)$$

Define $\xi \equiv p_t^* - x_t \cdot \theta^* = g(x_t \cdot \theta^*) - x_t \cdot \theta^*$. Writing $h''_t(p_t^*)$ in term of ξ , we obtain

$$h''_t(p_t^*) = \frac{1}{\sigma} \left[\frac{\xi}{\sigma} \left(\frac{1 - \Phi(\xi/\sigma)}{\phi(\xi/\sigma)} \right) - 2 \right] \phi(\xi/\sigma) \quad (65)$$

By tail bound inequality for Gaussian distribution $1 - \Phi(\xi/\sigma) \leq (\sigma/\xi)\phi(\xi/\sigma)$ for $\xi \geq 0$. Therefore,

$$\frac{\xi}{\sigma} \left(\frac{1 - \Phi(\xi/\sigma)}{\phi(\xi/\sigma)} \right) - 2 \leq -1, \quad (66)$$

and the same bound obviously holds for $\xi < 0$.

By definition of function g , $|\xi| \leq M$ with $M = W + \varphi^{-1}(0)$, and φ being the virtual valuation fusion corresponding to the Gaussian distribution (See Equation (11) and explanation following Equation (24)). Hence, $\phi(\xi/\sigma) \geq \phi(M/\sigma)$. Putting this together with (66), we get $h''_t(p_t^*) \leq -c_1$ with $c_1 = (1/\sigma)\phi(M/\sigma)$.

For the second part of the Lemma statement, set $\delta \leq \min \{M, \sigma^2/(6M), \sigma^2\phi(M/\sigma)\}$. For $p \in [p_t^* - \delta, p_t^* + \delta]$, we have

$$\left| \frac{p}{\sigma} \left(\frac{p - x_t \cdot \theta^*}{\sigma} \right) - \frac{p_t^*}{\sigma} \left(\frac{p_t^* - x_t \cdot \theta^*}{\sigma} \right) \right| \leq \frac{1}{\sigma^2} |p - p_t^*| \cdot |p + p_t^* - x_t \cdot \theta^*| \leq \frac{1}{\sigma^2} \delta (2M + \delta) \leq \frac{1}{2}. \quad (67)$$

Using Equation (66) we get $p(p - x_t \cdot \theta^*)/\sigma^2 \leq -1/2$. Further,

$$\left| \phi \left(\frac{p - x_t \cdot \theta^*}{\sigma} \right) - \phi \left(\frac{p_t^* - x_t \cdot \theta^*}{\sigma} \right) \right| \leq \frac{|p - p_t^*|}{2\sigma^2} \leq \frac{\delta}{2\sigma^2} \leq \frac{1}{2} \phi(M/\sigma). \quad (68)$$

Therefore,

$$\phi \left(\frac{p - x_t \cdot \theta^*}{\sigma} \right) \geq \phi(\xi/\sigma) - \frac{1}{2} \phi(M/\sigma) \geq \frac{1}{2} \phi(M/\sigma). \quad (69)$$

Combining (67), (69) we obtain

$$h''_t(p) = \frac{1}{\sigma} \left[\frac{p}{\sigma} \left(\frac{p - x_t \cdot \theta^*}{\sigma} \right) - 2 \right] \phi \left(\frac{p - x_t \cdot \theta^*}{\sigma} \right) \leq -\frac{1}{4\sigma} \phi(M/\sigma) = -\frac{c_1}{4}. \quad (70)$$

The result follows.

E.4. Proof of Lemma 6

Let $Z = x \cdot v$ and $\tilde{Z} = Z/\|v\|_2$. Note that $\text{Var}(\tilde{Z}) = 1$. Write the expectation in terms of the tail probability

$$\mathbb{E}(\min(Z^2, \delta^2)) = \int_0^{\delta^2} \mathbb{P}(Z^2 \geq t) dt = \int_0^{\delta^2} \mathbb{P} \left(|\tilde{Z}| \geq \frac{\sqrt{t}}{\|v\|_2} \right) dt. \quad (71)$$

We consider two cases:

- $\delta \leq \|v\|_2$: The right-hand side in (71) can be lower bounded as

$$\int_0^{\delta^2} \mathbb{P}\left(|\tilde{Z}| \geq \frac{\sqrt{t}}{\|v\|_2}\right) dt \geq \int_0^{\delta^2} \mathbb{P}\left(|\tilde{Z}| \geq \frac{\sqrt{t}}{\delta}\right) dt = 2\delta^2 \int_0^1 t \mathbb{P}(|\tilde{Z}| \geq t) dt \quad (72)$$

In the sequel, we provide two separate lower bounds for the right-hand side.

Let $\xi \equiv \mathbb{P}(|\tilde{Z}| \geq 1)$. We have

$$\int_0^1 t \mathbb{P}(|\tilde{Z}| \geq t) dt \geq \int_0^1 t \xi dt \geq \frac{\xi}{2}. \quad (73)$$

We proceed to obtain another bound which utilizes the fact $\text{Var}(\tilde{Z}) = 1$.

$$\int_0^1 t \mathbb{P}(|\tilde{Z}| \geq t) dt = \int_0^\infty t \mathbb{P}(|\tilde{Z}| \geq t) dt - \int_1^\infty t \mathbb{P}(|\tilde{Z}| \geq t) dt = \frac{1}{2} \text{Var}(\tilde{Z}) - \int_1^\infty t \mathbb{P}(|\tilde{Z}| \geq t) dt \quad (74)$$

For $t \geq 1$, we have $\mathbb{P}(|\tilde{Z}| \geq t) \leq \xi$. Further, by applying Chernoff bound, we get

$$\begin{aligned} \mathbb{P}(|\tilde{Z}| \geq t) &= 2\mathbb{P}(\tilde{Z} \geq t) = 2\mathbb{P}(e^{\lambda \tilde{Z}} \geq e^{\lambda t}) \leq e^{-\lambda t} \mathbb{E}(e^{\lambda \tilde{Z}}) \\ &= 2e^{-\lambda t} \prod_{i=1}^d \left(\frac{e^{\lambda \frac{v_i}{\|v\|_2}} + e^{-\lambda \frac{v_i}{\|v\|_2}}}{2} \right) \leq 2e^{-\lambda t} \prod_{i=1}^d e^{\lambda^2 \frac{v_i^2}{2\|v\|_2^2}} = 2e^{\frac{\lambda^2}{2} - \lambda t}. \end{aligned}$$

Setting $\lambda = t$ leads to $\mathbb{P}(|\tilde{Z}| \geq t) \leq 2e^{-\frac{t^2}{2}}$. Combining these bounds into (73), we obtain

$$\int_0^1 t \mathbb{P}(|\tilde{Z}| \geq t) dt \geq \frac{1}{2} - \int_1^\infty t \min(2e^{-\frac{t^2}{2}}, \xi) dt = \frac{1-\xi}{2} - \xi \log\left(\frac{2}{\xi}\right) \quad (75)$$

We summarize bounds (73) and (75) as in

$$\int_0^1 t \mathbb{P}(|\tilde{Z}| \geq t) dt \geq \min_{\xi \in [0,1]} \max\left(\xi, \frac{1-\xi}{2} - \xi \log\left(\frac{2}{\xi}\right)\right) > 0.05 \quad (76)$$

Turning back to Equation (72), in this case we have

$$\int_0^{\delta^2} \mathbb{P}\left(|\tilde{Z}| \geq \frac{\sqrt{t}}{\|v\|_2}\right) dt \geq 0.1\delta^2 \quad (77)$$

- $\delta \geq \|v\|_2$: Similar to the previous case, the right-hand side in (71) can be lower bounded as

$$\int_0^{\delta^2} \mathbb{P}\left(|\tilde{Z}| \geq \frac{\sqrt{t}}{\|v\|_2}\right) dt \geq \int_0^{\|v\|_2^2} \mathbb{P}\left(|\tilde{Z}| \geq \frac{\sqrt{t}}{\|v\|_2}\right) dt \geq 0.1\|v\|_2^2 \quad (78)$$

The above two cases can be summarized as $\mathbb{E}(\min(Z^2, \delta^2)) \geq 0.1 \min(\|v\|_2^2, \delta^2)$.

E.5. Proof of Lemma 7

We use a standard argument that relates minimax ℓ_2 -risk in terms of the error in multi-way hypothesis testing problem; See e.g. (Yang and Barron 1999, Yu 1997). Let $\{\tilde{\theta}_1, \dots, \tilde{\theta}_m\}$ be a δ -packing of set Ω , meaning that their pairwise distances are all at least δ . Parameter δ is free for now and its value will be determined later in the proof. We further let P_j denote the induced probability on market values $(v(x_1), \dots, v(x_T))$, conditional on (x_1, \dots, x_T) and for $\theta^* = \tilde{\theta}_j$. In other words, in defining distributions P_j we treat feature vectors fixed. Let ν be random variable uniformly distributed on the hypothesis set $\{1, 2, \dots, m\}$ which indicates the index of the true parameter, i.e., $\nu = j$ means $\theta^* = \tilde{\theta}_j$.

Define $d(\theta_1^T, \theta) \equiv \sum_{t=1}^T \min(\|\theta_t - \theta\|_2^2, C)$ and let μ be the value of j for which $d(\theta_1^T, \tilde{\theta}_j)$ is a minimum. Suppose that δ is chosen such that $\delta^2 \leq C$. If $d(\theta_1^T, \tilde{\theta}_j) < \delta^2 T/4$ then $\mu = j$, because assuming otherwise, we have $\mu = j' \neq j$, and by triangle inequality

$$\min(\|\tilde{\theta}_{j'} - \tilde{\theta}_j\|_2^2, C) \leq \min(2\|\theta_t - \tilde{\theta}_j\|_2^2 + 2\|\theta_t - \tilde{\theta}_{j'}\|_2^2, C) \leq \min(2\|\theta_t - \tilde{\theta}_j\|_2^2, C) + \min(2\|\theta_t - \tilde{\theta}_{j'}\|_2^2, C), \quad (79)$$

for all t , where we used the inequality $\min(a+b, c) \leq \min(a, c) + \min(b, c)$ for $a, b, c \geq 0$. Summing over $t = 1, 2, \dots, T$, we get

$$T \min(\|\tilde{\theta}_{j'} - \tilde{\theta}_j\|_2^2, C) \leq 2d(\theta_1^T, \tilde{\theta}_j) + 2d(\theta_1^T, \tilde{\theta}_{j'}) \leq 4d(\theta_1^T, \tilde{\theta}_j) < \delta^2 T,$$

where we used the assumption $\mu = j'$. But this is a contradiction because $\|\tilde{\theta}_{j'} - \tilde{\theta}_j\|_2 \geq \delta$ (they form a δ -packing of Ω) and $\delta^2 \leq C$.

Using Markov inequality, we can write

$$\begin{aligned} \max_j \mathbb{E}_{P_j} d(\theta_1^T, \tilde{\theta}_j) &\geq \frac{\delta^2 T}{4} \max_j \mathbb{P}\left(d(\theta_1^T, \tilde{\theta}_j) \geq \frac{\delta^2 T}{4} \mid \nu = j\right) \\ &\geq \frac{\delta^2 T}{4m} \sum_{j=1}^m \mathbb{P}(\mu \neq j \mid \nu = j) = \frac{\delta^2 T}{4} \mathbb{P}(\mu \neq \nu). \end{aligned} \quad (80)$$

We use Fano's inequality to lower bound the error probability on the right-hand side. We first construct a δ -packing of Ω similar to the one proposed in (Raskutti et al. 2011, proof of Theorem 1). Define

$$\mathcal{A} = \{q \in \{-1, 0, 1\}^d : \|q\|_0 = s_0\}.$$

As proved in (Raskutti et al. 2011, Lemma 5), there exists a subset $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ of cardinality $|\tilde{\mathcal{A}}| \geq \exp(\frac{s_0}{2} \log \frac{d-s_0}{s_0/2})$ such that the Hamming distance between any two elements in $\tilde{\mathcal{A}}$ is at least $s_0/2$. Next, consider the set $\sqrt{\frac{2}{s_0}} \delta \tilde{\mathcal{A}}$ for some $\delta \leq W/\sqrt{2s_0}$, whose exact value to be determined later. Then, for q in this set, $\|q\|_1 = \sqrt{2s_0} \delta \leq W$ and hence $\sqrt{\frac{2}{s_0}} \delta \tilde{\mathcal{A}} \subseteq \Omega$. Further, for $q, q' \in \sqrt{\frac{2}{s_0}} \delta \tilde{\mathcal{A}}$, we have the following bounds:

$$\|q - q'\|_2^2 \geq \delta^2, \quad (81)$$

$$\|q - q'\|_2^2 \leq 8\delta^2. \quad (82)$$

By (81), the set $\sqrt{\frac{2}{s_0}} \delta \tilde{\mathcal{A}}$ forms a δ -packing for Ω with size $|\tilde{\mathcal{A}}|$.

We now turn back to bound (80). Left-hand side can be lower bounded using Fano's inequality. We omit the details here as it is a standard argument and instead we refer to (Raskutti et al. 2011, proof of Theorem 1) for details. Using Fano's inequality and bound (82), we get

$$\mathbb{P}(\mu \neq \nu) = 1 - \frac{\frac{8T}{2\sigma^2} \delta^2 + \log(2)}{\frac{s_0}{2} \log \frac{d-s_0}{s_0/2}}. \quad (83)$$

Choosing $\delta^2 \leq \delta_1^2 \equiv \frac{\sigma^2 s_0}{32T} \log \frac{d-s_0}{s_0/2}$, we obtain $\mathbb{P}(\mu \neq \nu) \geq 1/4$. Therefore, setting $\delta^2 = \min(\frac{W^2}{2s_0}, \delta_1^2, C)$ and combining with bound (80), we conclude that

$$\min_{\theta_1^T} \max_{\theta^* \in \Omega} \mathbb{E}(d(\theta_1^T, \theta^*)) \geq \frac{\delta^2 T}{16} = \frac{1}{16} \min \left\{ \frac{W^2 T}{2s_0}, \frac{\sigma^2 s_0}{32} \log \frac{d-s_0}{s_0/2}, CT \right\}. \quad (84)$$

We derive another separate lower bound for minimax risk, by assuming that an oracle gives us the true support of θ^* . In this case, the least square estimator, applied to the observed features restricted to the true support S , achieves the optimal minimax ℓ_2 rate. This implies that $\|\theta_t - \theta^*\|_2^2 \geq c\sigma^2 s_0/t$, for $t \geq s_0$ and a constant $c > 0$. Therefore,

$$\min_{\theta_1^T} \max_{\theta^* \in \Omega} \mathbb{E}(d(\theta_1^T, \theta^*)) \geq \sum_{t=1}^T \min \left(c\sigma^2 \frac{s_0}{t}, C \right) \geq c' s_0 \log(T/s_0), \quad (85)$$

for some constant $c' > 0$, depending on C . Combining bounds in (84) and (85) and after some calculus, we obtain

$$\min_{\theta_1^T} \max_{\theta^* \in \Omega} \mathbb{E}(d(\theta_1^T, \theta^*)) \geq \tilde{C} \min \left\{ \frac{T}{s_0} + s_0 \log \left(\frac{T}{s_0} \right), s_0 \log \left(\frac{dT}{s_0^2} \right) \right\}, \quad (86)$$

for a constant \tilde{C} that depends on C, σ, W . The proof is complete.

Bibliography

- Shipra Agrawal and Nikhil R. Devanur. Bandits with concave rewards and convex knapsacks. In *Proceedings of the Fifteenth ACM Conference on Economics and Computation*, EC '14, pages 989–1006, 2014.
- Albert Ai, Alex Lapanowski, Yaniv Plan, and Roman Vershynin. One-bit compressed sensing with non-gaussian measurements. *Linear Algebra and its Applications*, 441:222–239, 2014.
- Airbnb Documentation. Smart pricing: Set prices based on demand. <https://www.airbnb.com/help/article/1168/smart-pricing--set-prices-based-on-demand>, 2015.
- Kareem Amin, Afshin Rostamizadeh, and Umar Syed. Repeated contextual auctions with strategic buyers. In *Advances in Neural Information Processing Systems*, pages 622–630, 2014.
- Victor F Araman and René Caldentey. Dynamic pricing for nonperishable products with demand learning. *Operations research*, 57(5):1169–1188, 2009.
- Moshe Babaioff, Shaddin Dughmi, Robert Kleinberg, and Aleksandrs Slivkins. Dynamic pricing with limited supply. In *Proceedings of the 13th ACM Conference on Electronic Commerce*, EC '12, pages 74–91, 2012. ISBN 978-1-4503-1415-2.
- Ashwinkumar Badanidiyuru, Robert Kleinberg, and Aleksandrs Slivkins. Bandits with knapsacks. In *Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on*, pages 207–216. IEEE, 2013.
- Mark Bagnoli and Ted Bergstrom. Log-concave probability and its applications. *Economic theory*, 26(2): 445–469, 2005.
- Hamsa Bastani and Mohsen Bayati. Online decision-making with high-dimensional covariates. Working Paper, 2016.
- Omar Besbes and Assaf Zeevi. Dynamic pricing without knowing the demand function: risk bounds and near-optimal algorithms. *Operations Research*, 57:1407–1420, 2009.
- Sonia A Bhaskar and Adel Javanmard. 1-bit matrix completion under exact low-rank constraint. In *Information Sciences and Systems (CISS), 2015 49th Annual Conference on*, pages 1–6. IEEE, 2015.
- Peter J Bickel, Ya'acov Ritov, and Alexandre B Tsybakov. Simultaneous analysis of lasso and dantzig selector. *The Annals of Statistics*, pages 1705–1732, 2009.
- Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.
- Josef Broder and Paat Rusmevichientong. Dynamic pricing under a general parametric choice model. *Operations Research*, 60(4):965–980, 2012.
- Peter Bühlmann and Sara van de Geer. *Statistics for high-dimensional data*. Springer-Verlag, 2011.
- Florentina Bunea et al. Honest variable selection in linear and logistic regression models via l_1 and $l_1 + l_2$ penalization. *Electronic Journal of Statistics*, 2:1153–1194, 2008.

- Emmanuel Candes and Terence Tao. The dantzig selector: Statistical estimation when p is much larger than n . *The Annals of Statistics*, pages 2313–2351, 2007.
- Emmanuel J Candes and Terence Tao. Decoding by linear programming. *IEEE transactions on information theory*, 51(12):4203–4215, 2005.
- Xi Chen, Zachary Owen, Clark Pixton, and David Simchi-Levi. A statistical learning approach to personalization in revenue management. Working Paper, 2015.
- Maxime C Cohen, Ilan Lobel, and Renato Paes Leme. Feature-based dynamic pricing. *ACM Conference on Economics and Computation*, 2016.
- A. V. den Boer and A. P. Zwart. Mean square convergence rates for maximum(quasi) likelihood estimation. *Stochastic systems*, 4:1 – 29, 2014. ISSN 1946-5238.
- Arnoud V den Boer. Dynamic pricing and learning: historical origins, current research, and new directions. *Surveys in operations research and management science*, 20(1):1–18, 2015.
- Arnoud V den Boer and Bert Zwart. Simultaneously learning and optimizing using controlled variance pricing. *Management Science*, 60(3):770–783, 2013.
- David L Donoho. Compressed sensing. *IEEE Transactions on information theory*, 52(4):1289–1306, 2006.
- Vivek F Farias and Benjamin Van Roy. Dynamic pricing with a prior on market response. *Operations Research*, 58(1):16–29, 2010.
- Vivek F Farias, Srikanth Jagabathula, and Devavrat Shah. A nonparametric approach to modeling choice with limited data. *Management Science*, 59(2):305–322, 2013.
- Alexander Goldenshluger and Assaf Zeevi. A linear response bandit problem. *Stochastic Systems*, 3(1): 230–261, 2013.
- Negin Golrezaei, Hamid Nazerzadeh, and Paat Rusmevichientong. Real-time optimization of personalized assortments. *Management Science*, 60(6):1532–1551, 2014.
- J Michael Harrison, Bora Keskin, and Assaf Zeevi. Bayesian dynamic pricing policies: Learning and earning under a binary prior distribution. *Management Science*, 58(3):570–586, 2012.
- Laurent Jacques, Jason Laska, Petros Boufounos, and Richard Baraniuk. Robust 1-bit compressive sensing via binary stable embeddings of sparse vectors. *IEEE Transactions on Information Theory*, 59(4): 2082–2102, 2013.
- Sham Kakade, Ohad Shamir, Karthik Sindharan, and Ambuj Tewari. Learning exponential families in high-dimensions: Strong convexity and sparsity. In *AISTATS*, pages 381–388, 2010.
- Nathan Kallus and Madeleine Udell. Dynamic assortment personalization in high dimensions. Working Paper, 2016.
- Godfrey Keller and Sven Rady. Optimal experimentation in a changing environment. *The review of economic studies*, 66(3):475–507, 1999.

- Bora Keskin. Optimal dynamic pricing with demand model uncertainty: A squared-coefficient-of-variation rule for learning and earning. Working Paper, 2014.
- N Bora Keskin and Assaf Zeevi. Dynamic pricing with an unknown demand model: Asymptotically optimal semi-myopic policies. *Operations Research*, 62(5):1142–1167, 2014.
- Robert Kleinberg and Tom Leighton. The value of knowing a demand curve: Bounds on regret for online posted-price auctions. In *Proceedings of 44th Annual IEEE Symposium on Foundations of Computer Science*, pages 594–605. IEEE, 2003.
- Nicolai Meinshausen and Bin Yu. Lasso-type recovery of sparse representations for high-dimensional data. *The Annals of Statistics*, pages 246–270, 2009.
- Roger B. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981.
- Sahand N. Negahban, Pradeep Ravikumar, Martin J. Wainwright, and Bin Yu. A unified framework for high-dimensional analysis of m -estimators with decomposable regularizers. *Statist. Sci.*, 27(4):538–557, 11 2012.
- Yaniv Plan and Roman Vershynin. One-bit compressed sensing by linear programming. *Communications on Pure and Applied Mathematics*, 66(8):1275–1297, 2013a.
- Yaniv Plan and Roman Vershynin. Robust 1-bit compressed sensing and sparse logistic regression: A convex programming approach. *IEEE Transactions on Information Theory*, 59(1):482–494, 2013b.
- Stanislav P Ponomarev. Submersions and preimages of sets of measure zero. *Siberian Mathematical Journal*, 28(1):153–163, 1987.
- Sheng Qiang and Mohsen Bayati. Dynamic pricing with demand covariates. Working Paper, 2016.
- Garvesh Raskutti, Martin J Wainwright, and Bin Yu. Minimax rates of estimation for high-dimensional linear regression over-balls. *IEEE Transactions on Information Theory*, 57(10):6976–6994, 2011.
- Pradeep Ravikumar, Martin J Wainwright, John D Lafferty, et al. High-dimensional ising model selection using l_1 -regularized logistic regression. *The Annals of Statistics*, 38(3):1287–1319, 2010.
- Michael Rothschild. A two-armed bandit theory of market pricing. *Journal of Economic Theory*, 9(2): 185–202, 1974.
- Mark Rudelson and Shuheng Zhou. Reconstruction from anisotropic random measurements. *IEEE Trans. on Inform. Theory*, 59(6):3434–3447, 2013.
- Sara A Van de Geer. High-dimensional generalized linear models and the lasso. *The Annals of Statistics*, pages 614–645, 2008.
- Sara A Van De Geer, Peter Bühlmann, et al. On the conditions used to prove oracle results for the lasso. *Electronic Journal of Statistics*, 3:1360–1392, 2009.
- Martin J Wainwright. Sharp thresholds for high-dimensional and noisy sparsity recovery using-constrained quadratic programming (lasso). *IEEE transactions on information theory*, 55(5):2183–2202, 2009.

- Zizhuo Wang, Shiming Deng, and Yinyu Ye. Close the gaps: A learning-while-doing algorithm for single-product revenue management problems. *Operations Research*, 62(2):318–331, 2014.
- Yuhong Yang and Andrew Barron. Information-theoretic determination of minimax rates of convergence. *Annals of Statistics*, pages 1564–1599, 1999.
- Bin Yu. Assouad, fano and le, cam. In *Research Papers in Probability and Statistics: Festschrift in Honor of Lucien Le Cam*, pages 423–435. Springer-Verlag, 1997.